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FREGE ON EXTENSIONS OF CONCEPTS, FROM 1884 TO 1903*

Tyler Burge

Russell's paradox defeated Frege's attempt to demonstrate logicism—the view that the mathematics of number can be derived from the axioms of logic together with definitions in logical terms of mathematically primitive expressions. The paradox indicated that Frege's logic was inconsistent and that the notion in terms of which he tried to define the cardinal numbers was defective. The defective notion, that of the extension of a concept, has remained interesting because it is motivated by intuitions that, arguably, play an ineliminable but inadequately understood role in modern set theories.[1] The development of Frege's views is also historically interesting. Frege was uncertain about the crucial notion and the paradox-producing axiom from the outset. He experimented with alternatives. What can be pieced together about his reasoning suggests deep tensions in his thought on fundamental matters.

This paper is purely historical. It concentrates on the period from the publication of *The Foundations of Arithmetic* in 1884 to the publication of *The Basic Laws of Arithmetic* (first volume, 1893; second, 1903). I shall trace the development during this period of Frege's notion of the extension of a concept.

I. Conceptual Uncertainty and Frege's Rationalism

The story begins with two remarkable passages in *The Foundations of Arithmetic*. Both are enigmatic. Both appear to express a lack

*I am indebted to the referee for suggestions that led to improvements.

[1]Cf. Charles Parsons "Some Remarks on Frege's Conception of Extension" in M. Schirn, ed. *Studien Zu Frege,* Stuttgart, Frommen-Holzboog (1976); "Sets and Classes" *Noûs* 8 (1974): pp. 1–12; "What is the Iterative Conception of Set?" In *Logic, Foundations of Mathematics, and Compatibility Theory,* Butts and Hintikka, eds. (Dordrecht: D. Reidel, 1977). The former article discusses Frege's notion of extension in the period from 1902, the discovery of the paradox, to 1925, Frege's death.

of commitment to the notion of the extension of a concept (*Umfang eines Begriffes*, or *Begriffsumfang*). The first occurs in a footnote to the passage in which Frege first defines the expression 'the Number that belongs to the concept F'. He defines the expression as: the extension of the concept *numerically equivalent to the concept F*. This is Frege's first use of the notion of the extension of a concept. And he immediately attaches the following famous footnote:

> (A) I believe that for "extension of the concept" we could write simply "concept." But two objections could be raised:
>
> 1. that this contradicts my previous statement that the individual numbers are objects, as is indicated by means of the definite article in expressions like "the number two" and by the impossibility of speaking of ones, twos, etc. in the plural, and also by the fact that the number makes up only an element in the predicate of a statement of number.
> 2. that concepts can have the same extensions, without coinciding (*zusammenfallen*).
>
> I am however of the opinion that both objections can be refuted; but here that would lead us too far afield. I assume that it is known what the extension of a concept is (*FA*, p. 80n).[2]

The second striking passage occurs in the summation of the book. After reviewing the difficulty of providing an explicit defini-

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[2]Textual references occur in the text. The following works of Frege will be cited with the abbreviations noted: *The Basic Laws of Arithmetic* (*BL*) Furth trans., (Berkeley: University of California Press, 1967); *Begriffschrift* (*B*) Angelelli, ed., (Hildesheim; Georg Olms, 1964); *The Foundations of Arithmetic* (*FA*), Austin, trans., (Evanston, Ill.: Northwestern University Press, 1968), (the pagination in the translation is the same as that in the original German); *Grundgesetze Der Arithmetik* (*GGA*) (Hildesheim, Georg Olms, 1962); *Kleine Schriften* (*KS*), Angelelli, ed., (Hildesheim, Georg Olms, 1967); *Nachgelassene Schriften* (*NS*), Hermes Kambartel, Kaulbach, eds., (Hamburg: Felix Meiner, 1969); *Philosophical and Mathematical Correspondence* (*PMC*), Kaal, trans., (Chicago, The University of Chicago Press, 1980); *Posthumous Writings* (*PPW*), Long and White, trans., (Chicago, The University of Chicago Press, 1979); *Translations from the Philosophical Writings of Gottlob Frege* (*G&B*), Geach and Black, eds., (Oxford, Basil Blackwell, 1966); *Wissenschaftlicher Briefwechsel* (*WB*), Hermes et al., eds., (Hamburg, Felix Meiner, 1976). Translations are mine. Changes from the excellent translations by Austin, Furth, and others are usually minor, though especially in Austin's case, they often affect the tone and are more literal.

tion for numerical expressions and citing the above-mentioned definition, Frege writes:

> (B) In this we take for granted the sense of the expression "extension of the concept." This way of overcoming the difficulty will not win universal applause, and many will prefer to remove the doubt in question in another way. I attach no decisive importance to bringing in the extension of a concept (*FA*, 117).

Several commentators have called attention to these passages. Most simply note the fact that Frege is uncharacteristically obscure and indefinite, especially given the fundamental importance of the definition in question. Some have misread the passages. I think that we can come to understand them by giving them a context and by reflecting on other writings that make reference to them.

The first question that arises concerns the nature of Frege's ambivalence. Is Frege indicating that he thinks that there are alternative, significantly different, but equally good definitions within logic of 'cardinal number'?[3] I think that the answer to this question is "no." Rather, Frege was uncertain both about his whole approach to the problem through explicit definition (B), and, granted that approach, about the use and explication of 'extension of a concept' (A). The full evidence for this answer must emerge slowly.

An initial and weighty consideration against the view that in (A) and (B) Frege was indicating that there are equally good non-equivalent ways of defining the numbers within his logic is simply a matter of common sense. Frege was starved for recognition after the poor reception of his *Begriffschrift* in 1879. If he had had in mind fundamentally different non-equivalent ways of defining number he would have announced the view, displayed the definitions, and explained the alternatives in some detail. In fact, he suggests one definition, but is lame in his remarks about it. The

[3]An affirmative answer is proposed by Paul Benacerraf in "Frege: The Last Logicist" *Midwest Studies* 6 (1981): pp. 29ff. Benacerraf claims that Frege's account of number was much more similar to modern accounts, in which alternative definitions of numbers within set theory are allowed, than is usually thought. He claims that (A) and (B) indicate that Frege did not expect even reference to be preserved by his definitions. Benacerraf's argument partly rests on a further passage in *The Basic Laws of Arithmetic* (section 10). I discuss this passage in "Frege on Truth," forthcoming, *Synthese*.

posture contrasts sharply with the confident, almost overbearing, tone of the rest of the book. It suggests uncertainty, not confidence in a plurality of ways of reaching his goal.

A larger reason for taking (A) and (B) to evince uncertainty derives from considering Frege's project from Frege's perspective. Frege regarded the *Sinn* and *Bedeutung* of 'Number' as not completely determined by conventional mathematical usage, at the time of his attempt to define it in his logic.[4] That is, the conventional significance of the term did not fix its *Sinn* or *Bedeutung*. Sense and conventional significance diverge elsewhere in Frege's philosophy—in his views on indexicality, for example.[5] The reasons for the divergence in this case are that mathematical language is logically or structurally unperspicuous, and that mathematical usage is, by Frege's standards, vague. For our purposes, vagueness is the key problem. To be fully competent by conventional standards one did not have to grasp something that settled all the questions about the application of the term that a genuine sense had to settle.

Thus, from Frege's point of view, "grasp" of the conventional, mathematical significance of 'Number' might be "complete" (one might count as fully understanding its current usage), while one's grasp of the term's sense was incomplete and uncertain. Frege held that senses (or "concepts"—which are in his earlier work an amalgam of sense and denotation, and later just denotation) are not themselves vague or indefinite. He held them to be eternal and absolutely determinate in their corresponding *Bedeutungen* (or in the objects that fall under them). The vagueness problem lay with human understanding and with conventional usage, not with the senses, or concepts, or their expression.

Strictly speaking, from Frege's point of view, no one had fully grasped and mastered the concept of number or the *sense* (as distinguished from the conventional significance) of 'Number' by the time he wrote *Foundations* in 1884. By scrutinizing passages (A) and

[4]Frege had not distinguished *Sinn* and *Bedeutung* in 1884, but the point would apply to both. The term 'Begriff' ('concept') had also not acquired the technical use it later acquired in Frege's work.

[5]Cf. my "Sinning Against Frege," *The Philosophical Review* 88, no. 3 (1979); pp. 398–432.

(B), I think we will find that Frege would have had to apply this point even to himself.

I will not have the space to justify this interpretation in depth. But I will discuss some of the passages that support it. In *Begriffschrift* section 27 Frege indicates that a vague expression does not express something "judgeable." This sort of remark is faintly echoed only once later in his career (*PMC* 114–115/*WB* 183). But he frequently states that vague concept expressions lack *Bedeutung* (e.g., *PPW* 122, 155, 179/*NS* 133, 168, 193–194; *PMC* 114/*WB* 183; *KS* 122–123). Now the sense for 'Number' that Frege sought to define would certainly fix a definite *Bedeutung*. So if 'Number' in its conventional usage is vague, its conventional understanding and explication will not articulate the concept denoted by the term, or the sense that it expresses (cf. GGA II, 69–70).

In several passages Frege implies that the conventional usage of such expressions as 'Number' *is* vague—not merely unperspicuous. At the end of *Foundations* (*FA*, 110), he indicates that the "meaning" (*Bedeutung*) (conventionally) assigned to 'sum' and 'product' before the introduction or discovery of complex numbers said nothing about the admissibility or inadmissibility of certain further assignments of "meaning." (Cf. *PMC* 63/*BW* 96; *G&B* 28/*KS* 131.) Nothing in then current mathematical usage determined how to extend the conventional "meaning" of 'number' to cover cases about which it was previously mum. Further philosophical and mathematical considerations had to be appealed to.

Foundations can be read as part of an attempt to produce the final in a series of sharpenings of the "meaning" of 'Number'. Referring, in the introduction, to the task of understanding the concept of number, Frege writes,

> What is known as the history of concepts is a history either of our knowledge of concepts or of the meanings of words. Often it is only through great intellectual labor, which can continue over centuries, that a concept is known in its purity, and stripped of foreign covering that hid it from the eye of the intellect (*FA* vii).

Seven years later, Frege embroiders this passage:

> For the logical concept, there is no development, no history. . . . If instead of [this sort of talk] one said "history of the attempt to grasp a

> concept" or "history of the grasp of a concept," it would seem to me far more appropriate; for the concept is something objective that we do not form and is not formed in us, but that we try to grasp and finally, it is hoped, really grasp—if we have not mistakenly sought something where there is nothing (*KS* 122).

Clothing or covering is Frege's metaphor for a concept's (or a sense's) verbal expression. The implication is that mastering conventional mathematical usage of the term 'Number' does not suffice for grasping the concept of logic that his labor seeks to reveal. Note also the implication in *FA* vii that there is a single "pure" concept to be found once one's intellectual labor is rewarded. Both passages highlight the difficulty of grasping a logical concept.

Section one of *Foundations* suggests that mathematical concepts such as function, continuity, limit, infinity stand in need of sharper determination. (Lack of sharpness is cognate for Frege with vagueness.) The ideal of "sharp grasp of concepts" is said to be the fundamental aim of the book. (Austin's translation obscures this point slightly by translating "scharf zu fassen" as "sharp definition.") Section 2 indicates that the concept of number is Frege's prime target in the attempt to attain "sharp grasp." In the light of these passages, the remarks about fixing a "sharply bounded" concept of number in *FA* 74, 79 should be seen not only as remarks about the particular stage of Frege's own inquiry, but also about the status of conventional understanding in mathematical science. In *Foundations* 81, he acknowledges that the remark that one number is wider than another, though supported by his definitions, is neither supported nor opposed by ordinary usage (*FA,* 81). As noted, the last pages of the book return to the vagueness theme. The discussion of complex numbers provides a model for his own method (*FA* 110ff): philosophical and new mathematical considerations must supplement conventional usage.

Articulation of the commonness of incomplete understanding recurs in Frege's discussion of "analytic" definitions in an unpublished work from 1914. I know of nothing that prevents seeing his remarks there as congenial with his earlier views. He says that in analyzing the sense of an expression with long-established usage, one may be uncertain whether the *analysans* gives the sense of the older expression. In such cases, he says, one can drop the older sign from the formal system and introduce a new counterpart whose sense is then "stipulatively" defined within the formal system. He continues:

> (C) How is it possible, one can ask, that it be doubtful whether a simple sign has the same sense as a complex expression, if not only the sense of the simple sign is known, but also the sense of the complex expression can be recognized from the way it is put together? In fact, if the sense of the simple sign is really clearly grasped, then it can not be doubtful whether it agrees with the sense of the complex expression. If this is open to question even though the sense of the complex expression can be clearly recognized from the way it is put together, the reason must lie in the fact that the sense of the simple sign is not clearly grasped, but appears as through a mist only with blurred outlines. (PPW 210–211/*NS* 227–228. Cf. also *PPW* 207/*NS* 224.)

The mist metaphor is closely associated with the metaphors of clothing or veiling, which are used to indicate that ordinary linguistic usage sometimes prevents a clear grasp of the associated concepts or senses (*FA* vii). The reason for an incomplete grasp of a mathematical concept or sense, in Frege's view, lay in the logical insufficiencies and vagueness of ordinary mathematical usage. These drawbacks were inseparable, in Frege's view, from insufficient foundational knowledge.

The project of defining 'Number' in purely logical terms cannot then be seen as attempting to specify its significance in any ordinary sense of 'significance'. For the sense and *Bedeutung* of 'Number' are not its conventional significance, and are only constrained—not fixed—by standard mathematical understanding and usage. On the other hand, arriving at a definition of 'Number' is not a matter of stipulation (except as seen within the confines of the formal system). Mathematical practice placed substantial restrictions on a good definition. Moreover, I think that any room for arbitrariness in attaching word to concept, or to sense, that survived these restrictions was conceived by Frege as arbitrariness relative only to conventional, mathematical usage. Frege was not interested in making just any mathematical model of arithmetic. The very project is foreign to his methodology. I think that he sought a single objective conception of number that underlay and fully rationalized mathematical practice (cf. note 3: Burge, forthcoming).

These considerations raise a question about the nature of sense expression (and concept denotation). How could the term 'Number' indicate a definite "concept" when all current mathematical understanding and usage failed to determine a sense or concept? Even granted that senses and concepts themselves are independent of human minds or human activity, how can Frege regard sense-*expression* (or the denotation of concepts by language) as even partly independent?

There is scattered evidence that Frege was aware of the force of these questions. For example, as we have noted, in *Begriffschrift* section 27 he writes of a particular vague expression that it does not have a judgable content. In 1880-81 he writes that the logical relations implicit in a particular logically unperspicuous expression *are not expressed,* but must be guessed at (*PPW* 13/*NS* 14). In a letter to Peano in 1896 he seems to indicate that vague expressions lack sense as well as *Bedeutung* (*PMC* 114–115/*WB* 183). And Frege repeatedly states that vague concept words lack *Bedeutung.* But these remarks cannot, without qualification, be pressed to yield their natural implication. Since the use and understanding of nearly all expressions are vague and unperspicuous by his standards, the natural implication is that nearly all expressions lack a sense and denotation. But this is not the way Frege usually writes. In numerous places, he writes without qualification of the concept of *number* and of the sense and denotation of expressions in ordinary language and mathematics.

One could, of course, distinguish between a full-strength doctrine—that vague expressions do not fully express senses—and an approximate doctrine for public consumption—that expressions express senses (*sotto voce:* if only indefinitely or approximately). There is something to this point. But it still leaves Frege with the unexplicated notion of indefinite expression.

I think that Frege was guided by a more general conception, one that is substantially different from the ordinary notion of conventional meaning-expression. According to the ordinary notion of expression, vague terms do not "express" definite senses or denote definite concepts. But this is just a consequence of the point that vague understanding and usage do not fix or articulate definite senses or concepts. To say, as Frege says, that 'Number' *does* denote a concept and *does* express a sense is to say that the ultimate foundation and *justification* of mathematical practice supplements current usage and understanding of the term in such a way as to attach it to a concept and a sense. From this point of view, vague usage and understanding do not entail vague sense-expression.

Currently, we may have to guess at the sense (*PPW* 13/NS 14). Even the best mathematicians may grasp it only haphazardly and intermittently (*PPW* 222/NS 240). Adequate, full grasp of what is "expressed" requires not just reflection on one's usage, but mathematical work and acquisition of mathematical knowledge (*PPW* 33/*NS* 37; *KS* 369.)

In spite of these peculiar points, there are two features of Frege's rationalism that allow him to count vague mathematical terms as "expressing" definite senses and denoting definite denotations. First, mathematical practice is held to be founded on a deeper rationale than anyone has previously understood. (This is, I think, the fundamental target of Wittgenstein's work on the philosophy of mathematics.) Second, Frege is attracted to construing the most fundamental sort of *a priori* knowledge on the model of insight or understanding. Thus the mathematical work of his logicist project is supposed to yield and be justified by the acquisition of insight—understanding of the true sense and denotation underlying ordinary mathematical uses of 'number' and appreciation of the self-evident, underlying principles of logic.

I cannot discuss the pros and cons of these features of Frege's rationalism here. There is no question that his deeply rationalist conception of sense-expression is highly idealized, and rather far removed from our ordinary notions of meaning, understanding, and thought. But the strong idealization is fully intended. As Descartes and Plato did before him, Frege emphasizes how difficult it is to attain full insight—full understanding (*FA* vii; *KS* 122; *PPW* 12–13, 222/*NS* 16–17, 240). Such understanding is not separable from the deepest sort of knowledge (*KS* 122–124; *PPW* 33/*NS* 37; *KS* 369). We shall return to these points in Section VI.

The uncertainty expressed in passages (A) and (B) thus occurs within a larger rationale. Frege was not only uncertain or unclear about sharp boundaries for the pre-established usage of such terms as 'cardinal number'. He was also uncertain about the definition he gave for it in the *Foundations of Arithmetic*. Although a cursory reading of (A) and (B) suggests that Frege's uncertainty centered on matters of substance, it is a subtle question to what extent his worries were expositional and to what extent they were substantial.

II. Passage (B)—Contextual Definition

Since, despite their kinship, passages (A) and (B) are indicative of very different tendencies in Frege's thinking, we shall discuss them separately. First, (B). In a letter to Russell, July 28, 1902, after discovery of the paradox, Frege writes that he had long struggled against recognizing courses of values, classes, or extensions of concepts, but that he has found no other answer to the question of how we grasp logical objects (numbers) than that we grasp them as extensions of concepts (*PMC* 140–141/*WB* 223).

There is evidence that Frege did struggle against using the notion of the extension of a concept in defining number. A manuscript from the period immediately after the publication of the *Foundations of Arithmetic* had the stated purpose of defining cardinal number without extensions of concepts, according to a list of Frege's *Nachlass* compiled by Heinrich Scholz.[6] Unfortunately, this manuscript was among those destroyed during an American bombing raid in 1943. Scholz's very brief summary of the manuscript makes it difficult to see how Frege reasoned. Besides the abovementioned purpose of the manuscript, the summary merely mentions three parts of the manuscript with the following headings:

a) Is it necessary to grasp numerical equality as strict identity?;
b) Is it possible to define a judgeable content that contains $N_\tau F(\tau)$ [Frege's early expression for 'the number of F's'] in such a way that one says that the content may not change whenever F is replaced by G, as long as $F(\varrho) \underset{\varrho}{\smile} G(\varrho)$ [F and G are numerically equivalent];
c) (The difficulty of the definition of an object through a recognition judgment [explicit definition]).

[6]Cf. Albert Veroort, "Geschichte des wissenschaftlichen Nachlasses Gottlob Freges und seiner Edition. Mit einem Katalog des ursprunglichen Bestands der nachgelassene-Schriften Freges" in *Studien Zu Frege,* Vol. I, *op. cit.,* item 47, p. 95.

Scholz notes that b) contains discussions of the definition of object and an early discussion of the definition of a concept's extension.

This is not much to go on. It appears that in b) Frege was raising questions about the notion of object, as well as trying to define or explicate 'extension of a concept'. It also appears that he was considering a contextual definition of 'the number of F's' that would leave a containing sentence's "content" unchanged. Perhaps in a) he was attempting to interpret the equals sign in arithmetic not as identity but as numerical equivalence. This latter notion he had defined (in *Foundations*) for the general case, in second order logic. Perhaps in c) Frege was reconsidering *Foundations* sections 56–61, 66–68, 107, in which he asserts that numerical expressions are singular terms and denies that a contextual definition of numerical expressions will yield a concept of number with sharp boundaries. His reason is that such definitions do not define whether an expression designates a number if it occurs outside the favored sentential contexts.

The difficulty for an explicit definition of numerical expressions, on the other hand, was that on Frege's view it forces one to take logic to be committed to the existence of particular objects, whose existence can be known *a priori* and on purely conceptual grounds. (cf. *FA*, sections 3, 88–89, 104–105.) Given his exposure to Kantian themes, and his development of first and second order logic in 1879, Frege must have found this view unobvious, despite occasional, hopeful rhetoric to the contrary (*FA,* section 105). Moreover, his strictures on explicit definition required him to *demonstrate* the existence and uniqueness of any object for which a singular term was thus introduced (*FA* 106, 114–115; *BL* 11–12/*GGA* I, 14). Even to someone unacquainted with the paradoxes and the incompleteness theorems, carrying out this task for extensions of concepts, for the general case, must have seemed a technically daunting task. (There is some suggestion of this in 'Über Formale Theorien der Arithmetik' 1885, *KS,* 110–111.) It was a task, of course, that Frege failed to fulfill (cf. *BL,* section 31; appendix II; *PMC* 132/*WB* 213.) The proof for particular cases would depend on Frege's Basic Law (V), about which Frege expressed uncertainties before 1902, and which turned out to be inconsistent.

It seems reasonable then to conjecture that in the lost, post-*Foun-*

dations manuscript Frege was reconsidering the whole question of whether numbers were objects—whether numerals and expressions like 'the number of F's' were primitive singular terms in arithmetic and in counting. He seems to have been contemplating a contextual definition of such singular terms (roughly in the spirit of Russell's "no class" theory). Perhaps he also considered an account of the equals sign as indicating not identity but numerical equivalence among concepts.

Responding to Russell's paradox in an appendix to the second volume of *Basic Laws,* 1902, Frege considers an alternative that bears some resemblance to the one he appears to have been considering fifteen years earlier. Frege considers treating terms for extensions (hence for numbers) as syncategorematic. Terms for extensions would be pseudo-singular terms, with no denotation. This is one way of construing the idea of providing singular numerical expressions with merely contextual definitions within logic. It may reflect his earlier investigation. Frege rejects this alternative on the ground that it would not allow one to explicate quantification into the position of such singular terms: "thereby, the generality of arithmetical propositions would be lost." He also claims that it would be incomprehensible on this view how one could speak of a Number of classes (extensions) or a Number of Numbers (*BL* 129–130/*GGA* II, 255).[7]

Frege does not, in the appendix, consider recapturing the numbers as functions or concepts in the manner of second order arithmetic, with no appeal to classes or extensions. It is clear, however,

[7]In response to a suggestion of Russell in correspondence, he also considers taking extensions to be "improper objects," "not objects in the full sense," for which "the law of excluded middle fails to hold." The idea is to restrict the instantiation from quantified sentences to singular terms denoting extensions in accordance with rules governing levels, in the spirit of Russellian type theory. (*BL* 128–129/*GGA* II, 254–255. Cf. also letter to Russell 9/23/1902; *PMC* 145ff.) Frege rejects this alternative because of its complication and unintuitiveness. He particularly objects to stratifying identity. I think it unlikely that he would have considered this alternative seriously prior to discovering the paradoxes. A sort of type theory was proposed by Schröder before the discovery of the paradoxes. But this flowed from a conception of classes that was quite different from Frege's. Cf. Alonzo Church, "Schröder's Anticipation of the Theory of Types" *Erkenntnis* 10 (1976): pp. 407–411. Frege strongly rejected both Schröder's and Russell's type theories (*KS* 197–198; *BL* appendix II).

that this idea would run against Frege's grammatical grain. Frege seems to have remained committed to the view throughout the period that numbers were, if *anything*, the denotations of singular terms—objects. Concept expressions could never denote what singular expressions denoted (*FA*, section 51).

Scholz's notes on the lost manuscript are the only evidence I know of that Frege seriously considered a fundamentally different way of defining singular numerical expressions. It is not very strong evidence. Still, in my view, it is probable that he was considering the possibility of contextual definition even in writing *Foundations* and that this is the sort of alternative to "bringing in extensions of concepts" that he had in mind in passage (B) (*FA* 117). For in that passage he is specifically concerned with the difficulties of explicit definition.

III. Passage (A)—Concept and Object

We have more evidence for interpreting passage (A). Indeed, I think it possible to say with some confidence what is meant. Frege referred to the footnote on three other occasions. We shall discuss two of these in this section, saving the third for section IV. By scrutinizing these retrospective remarks, one can see that in (A) Frege was alluding not to a fundamentally different method of defining numerical expressions, but to uncertainty over using the term "extension of a concept," as opposed to another expression with substantially the same ontological implications.

(A) is discussed by Frege in "On Concept and Object" (1892). There he writes:

> (D) If we keep in mind that, in my way of speaking, expressions like "the concept F" signify not concepts but objects, Kerry's objections will already for the most part collapse. If he thinks . . . that I have identified concept and extension of a concept, he errs. I merely expressed my view [presumably in (A)] that in the expression "the Number that attaches to the concept F is the extension of the concept numerically equivalent with the concept F" one could replace the words "extension of the concept" by "concept." Notice carefully here that this word [the word "concept"] is combined with the definite article. Besides, this was only an incidental remark on which I based nothing (*G&B* 48/*KS* 72).

This passage immediately suggests how in 1884 Frege intended to answer the first of the two objections he raises in (A) to the substitution: the substitution does not contradict his earlier statement that Numbers are objects because "the concept *numerically equivalent with the concept F*" denotes an object not a concept.[8]

Which object is, of course, not immediately clear. But there is reason to think that Frege was not here contemplating serious ontological divergence from the definition in terms of extensions of concepts. The substitution is justified by an identity. In a draft of "On Concept and Object" published in the *Nachlass* (written between 1887 and 1891), in the course of a passage overlapping the end of (D), Frege comments again on (A):

> (E) Besides, this was only an incidental remark on which I based nothing, in order not to have to grapple with the doubts to which it could give rise. So Kerry's opposition to it does not at all bear on the core of my position.* (*Whether instead of the expression "the extension of the concept" one should simply say "the concept," I see as a question of expediency.) (*PPW* 106/*NS* 116).

The remark about expediency (couched, as it is, in the formal mode) suggests that the issue on Frege's mind in passage (A) was partly expositional—how best to present the definition of number that he proposed in *Foundations,* section 68. The remark also suggests what object Frege supposed the concept *horse* to be. It was supposed to be the extension of the concept *horse.*

Although in 1884 Frege probably thought that 'the concept F' and 'the extension of the concept F' denoted the same object, he probably had not decided what extensions of concepts (or even concepts) were. Not until later (1891 in "Function and Concept") did he introduce the notion of courses of values, with which he identified extensions of concepts (cf. *BL* 5–6/*GGA* I, 9–10). The expositional uncertainty expressed in (A) is not entirely separable from this ontological uncertainty.

[8]Benacerraf in *op. cit.* p. 30 and Terence Ward Bynum, "The Evolution of Frege's Logicism" *Studien Zu Frege, op. cit.,* p. 282, infer from (A) that Frege was contemplating replacing extensions of concepts with concepts.

IV. Passage (A)—Ontological Options

To understand Frege's intellectual situation in 1884, one must remember that he was faced with a wide variety of uses, among respected authors, of such terms as 'extension', 'concept', 'class', 'set', 'system'. Not only usage, but conception varied. The variations were symptomatic of lack of clarity. Explications of the notions of set, even by such men as Cantor and Dedekind, now strike one as remarkably vague and unstandardized. It was a time of conceptual ferment.

More specifically, Frege was confronted, on one hand, by an ancient logical tradition of using 'extension' in discussions of natural language, and, on the other, by a newer tradition among mathematicians (represented in Frege's thinking chiefly by Cantor, Schröder, and Dedekind) that was attempting to define numbers in terms of sets (classes, systems). I will discuss these two traditions briefly.

It is likely that Frege was influenced in his use of the term 'extension' ('*Umfang*') by the Boolean logical tradition which, in turn, derives from Leibniz. Frege associates the term with Boole in unpublished work dating from 1880–1881 (*PPW* 15–16, 33/*NS* 16–17, 37; cf. *LW*, 52). The term is also used in Mill's *System of Logic* (*SL* 3–4) and Jevons's *The Principles of Science* (*PS* II, 2)—both of which Frege cites in *Foundations*. Extension of a concept (*Begriffsumfang 'Begriffsumfang'*) is also used by Schröder—interchangeably with 'class' (*Klasse*) (e.g., *AL* I, 222, 233—Schröder gives space to expounding Mill). Jevons, for example, writes ". . . every general name causes us to think of some one or more objects belonging to a class. A name is said to denote the object of thought to which it may be applied . . . the objects denoted form the extent of meaning of the term; the qualities form the intent of meaning" (*PS* II, 2).[9] Jevons's statement brings out the emphasis,

[9]George Boole, *Collected Logical Works* (*LW*), vol. II. (Chicago: The Open Court Publishing Company, 1940), (*the Laws of Thought* was published in 1854); John Stuart Mill, *A System of Logic* (*SL*) (London: Longmans, Green and Co., 1865); W. Stanley Jevons, *The Principles of Science* (*PS*) (London, Macmillan and Co., 1920)—first published, 1865; Ernst Schröder,

within this logical tradition, on a dual meaning for concept words. Each sort of meaning (property and extension) was indicated, in some sense, in every use of a general term.

The tradition tends to emphasize that the intension of a concept or term is "more basic" than its extension since the qualities, marks, or characteristics that form the intension define or fix the extension (and not vice-versa) and because the extension is thought only "through" such characteristics (cf. *LL,* 102; *SL,* v, 4; *PS* XXXI, 14). It is also sometimes argued, apparently in view of context-dependent constructions, that the intension is more basic because extension shifts though the intension remains constant (cf. *SL,* v, 3; *PS* III, 7). This is, of course, an argument that left Frege cold—perhaps because it is inconsistent with the view that intensions fix extensions. But the semantical and epistemic arguments for the priority of intensions must have struck a responsive chord.

This view of the semantical and epistemic priority of intensions—properties, qualities, characteristic marks—over classes was opposed, at least implicitly, by the development of set-theoretic approaches to number theory by authors like Cantor and Dedekind. There was a fair amount of contempt among mathematicians for traditional logic—particularly, for its lack of precision and its fruitlessness. Frege refers to this attitude occasionally (*PPW* 34/*NS* 38; *KS* 104). And Cantor exhibits it in his 1885 review of *The Foundations of Arithmetic* (*CGA* 400), with a disparaging reference to 'extension' and the old "school logic."

The beginnings of an analysis of number within intuitive set theory had been carried out at the time Frege wrote *Foundations.* Frege must have learned from this technical work. In fact, in 1885 Frege credits Cantor's 1885 review (and less explicitly in *FA* 97–98, Cantor's 1883 paper) with an equivalent definition of 'the number of F's' (*KS* 112; cf. *CGA,* 441, 167; *FA* 97–98).[10]

Vorlesungen über die Algebra der Logik (*AL*) Bd. I, (Bronx, N.Y.: Chelsea Publishing Company, 1966). The term 'extension' has a long history deriving from the Latin '*extensive*'. Sir William Hamilton, *Lectures on Logic* vol. II (New York: Sheldon and Company, 1858), pp. 100–101, traces the term through the *Port Royal Logic* (1718) to Cajetan (1496) back to Boethius. He thinks it has roots in Aristotle.

[10]Cantor's definition, in his 1885 review of Frege's *Foundations,* goes back to Cantor's *Grundlagen einer allgemeinen Mannigfaltigkeitslehre,* 1883.

Despite his likely respect for the mathematical work in this tradition, Frege was, from very early on, dissatisfied with its conceptual foundations. In particular, he criticized that notion of class or set that derives from grouping together elements—the ancestor of the iterative conception of set. He associates the notion with a "childish" Millian viewpoint, according to which the elements of a set are seen simply as physical parts of a physical whole (*KS* 105; *FA* sections 22–25, 28).[11] He points out that any conception of class or set that would be relevant to explicating what numbers are would have to include events, methods, and concepts. And he holds that these are not reasonably seen on the part-whole model (*ibid.*). He further believes that attempts to attain an appropriate level of abstraction, sufficient to deal with the wide applicability of the notion of number typically lead to psychologistic accounts of grouping the elements: accounts that appeal to the mind's abstracting from physical properties and the like (*FA*, sections 25–28, 50–52; *KS* 164, *PPW* 69–71/*NS* 77–80; *BL* 29–31/*GGA* I, 1–3). After 1884 he particularly indicts Dedekind and Cantor on this charge.[12]

Three of Frege's criticisms of the "element-grouping" conception of set are independent of his association of it with Mill's "agglomerative" conception and with psychologistic accretions. Frege repeatedly appeals to the criticisms in the period 1880–1895 (as well as later). The primary objection derives from the tradition in logic, mentioned earlier, according to which intensions (properties, marks, characteristics) are prior to extensions. More specifically, for Frege, predication is epistemically and logically prior to abstract

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Cf. Georg Cantor, *Gesammelte Abhandlungen* (*CGA*), Zermelo, ed., (Hildesheim, Georg Olms, 1962). In a note on the review, *CGA* 441–442, Zermelo says that the definitions are equivalent. Counting them so involves ignoring the deep differences between Cantor's and Frege's notion of class. Zermelo is aware of this.

[11]For a nonhistorical discussion of this notion, see my "A Theory of Aggregates" *Noûs* 11 (1977): pp. 97–117.

[12]There is good reason to think that Frege was right on the *ad hominem* point. Cantor's informal remarks about sets carry a remarkably strong, though probably not entirely intended, suggestion of dependence of sets on human minds (*CGA* 283, 387–388, 411, 413n, 440). Similar points apply to Dedekind's remarks about sets and numbers—except that Dedekind's psychologism seems more deeply meant. (Cf. Richard Dedekind, *Gesammelte Mathematische Werke* Bd. III, Fricke, Noether, Ore, eds. (Braunschweig: Freidr. Vieweg & Sohn Akt.-Ges., 1932), pp. 335, 344ff.)

objects. Frege's point is that the elements of a class are fixed, "delimited," only through "concepts." Elements are what fall under the concept. The relevant elements are determined, held together as a totality, only through "characteristics" they have in common—through a rule governing elementhood (*PPW* 34/*NS* 38; *FA* 59–67; *KS* 105, 164, 210; *BL* 30/*GGA* I, 2–3; *G&B* 105/*KS* 209). Frege explicitly associates concepts with properties and characteristic marks—both terms from the Boolean-Millian logical tradition.

I think that behind the objection lies Frege's assumption of a context principle according to which the justification for postulating abstract entities, such as sets, extensions, numbers, concepts and the like, derives from their role in providing denotations or meanings (*Bedeutungen*—in the earlier writings, contents, *Inhalte*) *for components* of *sentences* that express true thoughts. Frege held that one could think about such objects only through predication or nominalizations of predication; one could not justify the postulation of abstract entities by making lists. Such a procedure, he writes, is "very arbitrary and in actual thinking without significance" (*PPW* 34/*NS* 38). An evaluation of the context principle and Frege's objection is, however, beyond the bounds of this paper.

Two other objections are in effect corollaries of the first. One is that the element-grouping conception cannot account for the null class. By contrast, the idea that there are concepts under which no entities fall is familiar and natural (*PPW* 34/*NS* 38; *FA* 30, 59, 64; *KS*, 105: *G&B* 102/*KS* 206–207). The other objection is that the notion of an infinite, completed totality, even a totality with the cardinality of the first number class, could not be justified by "logical addition." Human beings are inevitably finite, and "inner intuition" is too subjectivistic and unclear a notion to use in justifying such totalities. "Going on in the same way," he might have added, is too indefinite a notion to explicate the objectivity and seeming mathematical definiteness of these large totalities. Such totalities could be justifiably seen only as deriving from definite concepts (rules of application) indicated in thought (*PPW* 34/*NS* 38; *KS* 164; *BL* 31; *GGA* I, 2–3; cf. also *FA* 98).

Frege was guided by these objections in insisting that any abstract, class-like objects in terms of which the numbers were to be defined had to be derivative from concepts, which he had regarded since 1879 not only as properties or characteristic marks, but also as a special sort of function (*G&B* 12–14/*BS* 16–17).[13] So extensions of concepts would have to be, in some sense, objects derivative from concepts. Yet they could not *be* concepts because of Frege's insistence that an object could not be predicative: concepts were associated only with predicates (or other function expressions); objects were denoted only by singular terms. I shall not attempt to explain this notorious insistence, though we shall return to it (cf. *G&B* 42–55/*KS* 167–178).

Bearing in mind Frege's attitudes to the logical and mathematical traditions, let us return to the question of why in (A) Frege considered using 'the concept *F*' as an alternative to 'the extension of the concept *F*', to denote the same thing.

One reason may have been that Frege saw 'the concept *F*' as the natural expression, within his logical theory, for the notion traditionally but confusingly expressed by "the concept in extension." Extensions were objects—the referents of singular terms—but they were also spoken of in the logical tradition as aspects or guises of concepts. Another reason might have been that the substitution would have been more palatable to a mathematical tradition suspicious of the term 'extension', but used to talking of functions in an extensional way (i.e., in such a way that functions were counted the same if for the same arguments they had the same values). A trace of this motivation may be seen in a long footnote to a discussion of Basic Law V in the second volume of *Basic Laws:*

> Few mathematicians will reflect upon expressing the circumstance that f( ) always has the same value as g( ) for the same argument as "f=g". Nevertheless, the error involved in this springs from an inadequate conception of the essence of the function. . . . Although accordingly the designation "f=g" cannot be recognized as correct, it shows that the mathematicians have already made use of the possibility of our transformation [in Basic Law V] (*GGA* II, 148).

Given Frege's long-standing inclination toward identifying concepts with certain functions, the substitution may have seemed expositionally attractive. The expression 'the concept . . .' itself was

[13]Hans Sluga, *Gottlob Frege* (London: Routlege & Kegan Paul, 1980), p. 81, emphasizes the point and does much else to show how many of Frege's deepest conceptions derive from the 1879 *Begriffschrift.*

sometimes used by Cantor, and no doubt others to refer to abstractions.[14] Frege may have seen in the expression a means of enforcing his insistence that classes were derivative from (unsaturated) concepts.

In fact, Cantor uses the expression in almost precisely the same context in which Frege used 'the extension of the concept . . .' in defining 'the number of F's' in *Foundations,* 79–80. Frege's third reference to passage (A) occurs in a reply to Cantor's uncomprehending 1885 review of *Foundations.* Cantor had criticized Frege for defining 'the number of F's' as the extension of *F* (*CGA* 440). Frege (1885) points out that this was not his definition. He makes reference to Cantor's definition of 'power' (*Mächtigkeit*), which for finite cardinal numbers was equivalent to Frege's definition of 'Number' ('*Anzahl*') (if one ignores Frege's special doctrines about what concepts are—cf. note 10 above). In effect, Frege alludes to the equivalence and adds:

> (F) Incidentally, the difference that Mr. Cantor writes 'General Concept' ('*Allgemeinbegriff*') where I write 'extension of the concept', seems inessential as noted in the footnote, p. 80 [i.e., passage (A)] in my work (*KS* 112).

Cantor's definition derives from a paper of 1883 that Frege cites in *Foundations.* It is thus possible that Frege's mention of the substitutability of 'the concept' for 'the extension of the concept' in *Foundations,* passage (A), was a not very explicit acknowledgement of the equivalence of Cantor's 1883 definition (cf. note 10; also *Foundations,* 97–98). The definition of number in terms of numerical equivalence was a key and original feature of Cantor's program, and it is possible that Frege got the idea from Cantor. Later, in 1892, Frege decided that Cantor's notion of class was too different from his own to count the definitions equivalent (*KS* 164).

The interchange with Cantor and the fact that in 1884 Frege

[14]Charles Parsons pointed out in conversation that Cantor consistently uses 'Begriff' for the cardinality of a set and 'Menge' or other terms for sets. One might speculate that Cantor had at least some sense of the modern view which demands provision for a distinction of levels. Frege's distinction between class and concept and his assimilation of classes to extensions of concepts seem to flow from an entirely different motivation.

already firmly distinguished concept and object suggest how Frege might have answered the second of the two objections he raises in (A) against his view that 'the concept' could replace 'the extension of the concept'. The objection was that concepts can have the same extension without coinciding. Two complementary answers were available to Frege. First, as Furth points out, Frege used 'coincide' ('*zusammenfallen*') equivalently with 'are identical' (*BL,* xiii–iv; *KS* 184). Since identity is a relation between objects, not concepts, concepts can indeed have the same extensions without themselves being identical. But then since 'the concept . . .' denotes an object—a class or extension, not a concept (*FA* 63)—the objection is irrelevant. Second, Frege may, by 1884, have been contemplating a mathematician's extensional view of concepts and functions: no discrimination of concepts without discrimination among objects that fall under them. Thus concepts with the same objects falling under them stand in the analog for concepts of the relation of identity. Concept expressions with the same objects falling under them are intersubstitutable *salve veritate.* At any rate this is an answer he would have given by 1891.[15]

Let us summarize the view of passage (A) that has so far emerged. Frege appears to have regarded the substitution of 'the concept' for 'the extension of the concept' as ontologically insignifi-

[15]There is some evidence that before *Foundations* Frege had adopted from the tradition in logic a more intensional view of "concepts": (1880–1881: *PPW* 18/*NS* 19–20). By 1891 at latest, he was propounding the extensional view, sometimes expressly noting its congeniality with the writings of "extensionalist logicians" (*G&B* 21/*KS* 125; *G&B* 80/*KS* 183–184; *PPW* 118ff/*NS* 128ff; *BL,* sections 1–4, 9–10, 36; *GGA,* II, 146–147). The evidence regarding Frege's position on the matter in *Foundations* is sketchy. But a number of considerations suggest that under pressure from mathematical considerations, Frege might have already found the extensionalist conception attractive. (Cf. (A), the exchange with Cantor (E), sections 63–65, and *PPW* 16/*NS* 17.)

The development of the *Sinn-Bedeutung* distinction after 1884 certainly made it easier for Frege to adopt an extensional view of concepts. Or perhaps the latter helped motivate the former. The *Sinn* of a concept word could fulfill the cognitive role that motivated intensionalist views of concepts, and *Bedeutung* could fulfill the mathematically and semantically fundamental role of contributing to truth-value. It would be interesting to know the exact developmental story about the relation between Frege's views on concepts and on *Sinn-Bedeutung.* (Cf. *PPW* 112–123/*NS* 133–134.)

cant. Both denoted objects. The terminological ambivalence reflects an uncertainty about how to present his logicist theory, but it also reflects unclarity about what numbers are. This unclarity stemmed partly from unclarity about what concepts were. Whereas it is debatable whether in 1884 Frege already viewed concepts extensionally, it is virtually certain that he did not yet view them as functions whose values are truth values (*BL* 5–6/*GG* I, 9–10). Without this view, he could not yet have developed his explication of what extensions of concepts were.

Frege was dissatisfied with the "element-grouping" conception of classes, but had not yet developed a satisfying alternative. In this context, he was exposed to Cantor's 1883 definition of 'power' (roughly, 'cardinal number'). It is unclear whether Frege developed his own definition independently, or whether his definition made use of Cantor's. At any rate, in writing *Foundations* he was confronted with a technically similar definition couched in a somewhat different vocabulary—a vocabulary that highlighted difficult issues about the relation between concept and object, and the very meaning of 'concept' and 'extension of a concept'. In passage (A), Frege notes the alternative formulation and indicates that he thinks the difference is not substantial. He contemplates maintaining the strict distinction between concept and objects, while claiming that 'the concept F' denotes an object—presumably the same object as does 'the extension of the concept F'. What this object was was to be decided by further work. As passage (B) suggests, Frege was not even sure that he wanted to stand by his appeal to objects at all, much less what objects "extensions of concepts" were to be.

V. Justifying Law (V)—Retreat from Passage (A)

Let us turn to the role in all this of Basic Law (V), the axiom governing extensions of concepts that led to contradiction. Frege's willingness to substitute 'the concept F' for 'the extension of the concept F' in passage (A) reflects more than an implicit acknowledgement of Cantor's different terminology in defining 'Number'. It also reflects Frege's struggle to justify Law (V) as a logical law. This struggle was inextricably bound up with the attempt to arrive at a satisfactory conception of logical objects—extensions of concepts.

The discussion of equivalence class definitions and the analogy between direction and number in *Foundations* (*FA*, sections 62–67) suggest that in 1884 Frege may have already had in mind the form of Law (V). In fact, he states the basic idea as early as 1880–1881, attributing it to the Boolean tradition (*PPW* 16/*NS* 17). The law can be formulated as follows:

$$\text{(V)}\ (x)(Ax \leftrightarrow Bx) \leftrightarrow \grave{\epsilon}A(\epsilon) = \grave{\alpha}B(\alpha)$$

(where '$\grave{\epsilon}A(\epsilon)$' may be read, 'the extension of the concept A'). In 'On Function and Concept', (1891) Frege claims that the two sides of an equivalence that is formally similar to (V) have the same *sense* (*G&B* 26–7/KS 130–1): '$(x)(x^2 - 4x = x.(x - 4))$' is said to have the same sense (*Sinn*) as '$\grave{\epsilon}(\epsilon^2 - 4\epsilon) = \grave{\alpha}(\alpha.(\alpha - 4))$'. In the same article Frege assimilates concepts to functions. So there is no deep difference between this remark about functions and an analogous remark about concepts. Moreover, this is the article in which Frege first introduces the *Sinn-Bedeutung* distinction. So it appears that Frege was conceiving at least instances of (V), and probably (V) itself, as expressing *the same sense* on the two sides of the biconditional. Such a conception, if correct, would certainly justify (V) as a logical law. For the biconditional would have the same sense and same self-evidence as '$(x)(Ax \leftrightarrow Bx) \leftrightarrow (x)(Ax \leftrightarrow Bx)$', and would not depend for its truth on any notions or truths in a special science (cf. *FA* 4; *BL* 127/*GGA* II, 253).

What led Frege to this implausible view? There seem to be two possible, mutually compatible, sources. One is the logical tradition, which we discussed earlier, in which each sentence containing a general term, and each general term, had a dual "meaning"—an "intensional" and an extensional one. Frege was clearly influenced by this tradition. He took his term 'extension of a concept' from it. Very probably his insistence on the priority of concepts (which he originally viewed as properties, probably interpreted intensionally) over classes was influenced by it. As he came to view concepts extensionally and developed the *Sinn-Bedeutung* distinction, he may have retained the view that, (even in *addition* to a sense), the use of concept words fixed a dual "meaning"—concept and extension of the concept (cf. *PPW* 122–123/*NS* 133–134). Although they "meant" (*bedeuteten*) only concepts, their use fixed an extension as well. In natural language, a given sentence with a given sense determined both *Bedeutungen* (cf. *G&B* 49–50/*KS* 173–174). In Frege's formal notation, distinct sentences would be used in denoting the distinct *Bedeutungen,* but they would express the same sense—only "in different ways" (*G&B* 27, *KS* 130; cf. *FA,* section 64).

The second possible source of motivation for seeing the two sides of Law (V) as expressing the same sense derives from Frege's use of 'the concept F'. As far back as 1879, Frege had fallen into the rather natural habit of reading sentences of the form F(*a*) as '*a* falls under the concept *F*' or G(*a,b*) as '*a* stands in the *G* relation to *b*' (*B,* section 10; *PMC* 101/*WB* 164–165; *FA,* sections 55, 58, 74–75; *G&B* 30, 46–47/*KS* 133, 171; *BL,* section 4). This habit continued through the publication of *Basic Laws.* It is clear that he takes the two modes of expression to have the same sense. Perhaps at first Frege regarded 'the concept F' (and 'falls under' in 'falls under the concept F') as syncategorematic. But passage (A) suggests that in 1884 he took 'the concept F' to be referential. This view is quite explicit in "On Concept and Object" (1892). Frege writes that 'the concept *man*' denotes an object and that in 'Jesus falls under the concept *man*', 'the concept man' is only part of the predicate. The whole expression 'falls under the concept man' means (*bedeutet*) the same as 'a man' (*G&B* 46–47/*KS* 171; cf. also *G&B* 45/*KS* 170). Similarly, at the end of the essay, Frege writes that in the sentence 'the number 2 falls under the concept *prime number*', 'falls under' expresses a sense that is two ways unsaturated and denotes (*bedeutet*) a relation (between objects) (*G&B* 54–55/*KS* 178).

It is immediately after the discussion of 'Jesus falls under the concept *man*' that Frege refers back to passage (A) in passage (D). He then claims that

(i) There is a square root of 4

and

(ii) The concept square root of 4 is realized

express the same thought. But he holds that the first six words of the (ii) denote an object, whereas 'is a square root of 4' in (i) denotes a concept. Similarly, the quantifier in the (i) does not denote a concept co-extensive with what is denoted by 'realized' in the (ii). (The former concept is second-level; the latter, first-level.) Frege then insists that a thought can be split up in many ways and that the same *sentence* (*Satz*) (as well as the thought it expresses) can be regarded as an assertion about a concept (*square root of 4*) and also an assertion about an object (the concept square root of 4)—"only we must note that what is asserted [assertively predicated] is different" (*G&B* 49/*KS* 173; cf. also *G&B* 50/*KS* 173–174). This remark sheds interesting light on Frege's conception of the context principle (or syndrome of principles) at this stage in his career. It is also reminiscent of the "dual meaning" aspect of extension-intension theory discussed earlier.[16]

In '2 falls under the concept *prime number*', 'falls under the concept *prime number*' denotes the same concept on Frege's theory of 1892 as does "$\cap\grave{\varepsilon}$Prime Number(ε)," since the expressions are co-extensive. This latter expression (defined in *BL* 34) could be roughly read 'falls in the extension of the concept *prime number*'.

[16]There is a noteworthy parallel between this doctrine of 'falls under' and Frege's doctrine of truth. In this same year (1892, 'Über Sinn und Bedeutung' *G&B* 63–64/*KS* 149–150), Frege holds that an attribution of truth to a thought does not express a different sense from the thought itself: the relation between a thought and truth is that of sense to *Bedeutung*. Thus (iii) p, and (iv) p = the True, express the same thought, even though an identity relation is denoted in (iv) but *seems* not to be in (iii). This can be compared to the doctrine of 'On Concept and Object' that (v) Fa, and (vi) *a* falls under the concept *F*, express the same thought even though a two-place relation is denoted in (vi) but seems not to be in (v). Apparently, according to the views of 'On Concept and Object' the same objects and concepts are, if not denoted, at least "fixed" in both (v) and (vi). Frege is not explicit about whether the point applies to (iii) and (iv), but I see no reason to think that he treated the cases differently.

This doctrine as applied to (v) and (vi) easily leads, by reasoning similar to that in *G&B* 54–55/*KS* 177–178, to the conclusion that an infinity of objects and concepts are "fixed" within any given sentence and any given thought. For it appears that on the doctrine, (vi) has the same sense as

(vii) *a* stands in the relation of falling under the concept *F* to *b*.

Here '_____ stands in _____ to _____' is a three-place relation among three objects. Unless some means of restricting new concepts can be produced, an infinity is generated. "Dual meaning" would be an understatement.

Here it would seem that 'falls in' is coextensive with 'falls under' and 'the extension of the concept *prime number*' denotes the same object as 'the concept *prime number*'.

So it appears that in 1891–1892, Frege was still working out the implications of the substitution of 'the concept . . .' for 'the extension of the concept . . .' contemplated in passage (A). I conjecture that he was at least considering the possibility that the ordinary language paraphrase of F*a* into "*a* falls under the concept *F*" would lend support to the view that Basic Law (V) expressed the same sense on each side of the biconditional. Expressions of the form 'the concept *F*' would denote the extensions or courses of values that Frege needed to define the numbers.

Sometime soon after the publication of "On Concept and Object" and "On Sense and Denotation," Frege rejected this line of thought. In "Comments on Sense and Denotation" (dated in the period 1892–1895), Frege recommends that expressions of the form 'the concept F' be rejected (*PPW* 122/*NS* 132–133) because "the definite article points to an object and belies the predicative nature of the concept." He goes on to state Law (V) in terms of extensions. He does not, however, deny that such expressions as 'the concept F' denote objects.[17] What he rejects is use of the expressions *in logic*. The reason he gives was fully available to him when he wrote "On Concept and Object." But there is no evidence until the present essay that he was willing to forego such use in logic. It is noteworthy that Frege's reason may be interpreted as centering on expositional considerations. The potential for misunderstanding that Frege had anticipated in (A) was realized in Kerry's objections. In any case, Frege gave up whatever hopes he may

[17]In fact, in a footnote on the same page, he says that they do: "These objects have the names 'the concept ϕ' and 'the concept X'." Earlier, however, in the same essay he may be expressing doubt: "the denotation of the expression "the concept *equilateral triangle* (insofar as one exists) is an object" (*PPW* 119–120/*NS* 130). Sluga, *op. cit.* pp. 142–143 claims that in this essay Frege "no longer holds ['the concept F'] to refer to anything." This claim is not supported by the evidence. Sluga also reads (A) as indicating that even when he thought the expression denoted an object, Frege never took the object to be the extension of the concept F. Sluga provides no support for this view. It is true that Frege never explicitly proposes the identification. But our evidence suggests that he made use of it, at least tentatively.

have had for using his views about 'the concept F' to strengthen the intuitive justification of Basic Law (V).

Although Frege uses 'the concept F' in an informal expositional way in *Basic Laws* (e.g., *BL,* section 3), there is no suggestion that the expression could be put to serious scientific use. Frege warns that it is misleading; he says that it does not denote a concept; and he omits to say that it denotes an object (*BL,* section 4, footnote). It is quite possible that by 1893 he regarded the expression as syncategorematic.

As far as I can find, however, he does not explicitly assert this view until several years after the discovery of the paradox. In a letter to Bertrand Russell of July 28, 1902 (*PMC* 141/*WB* 224), he says that the expression 'is a concept' is "logically speaking really to be rejected." He notes that it should name a second level concept [at least!], but it presents itself "linguistically" as a name of a first level concept. But even here, Frege does not explicitly say that 'the concept *F*' fails to denote. In 1906, he does so.

> Language stamps a concept as an object, since it can fit the designation of a concept in its grammatical framework only as a proper name. But it thereby really commits a falsification. Thus, the word 'concept' is itself strictly speaking already defective, in that the words 'is a concept' demand a proper name as grammatical subject; for they thereby demand a contradiction, since no proper name can designate a concept; or perhaps better still a piece of nonsense (*Unsinn*). . . . In the sentence 'Two is a *prime number*', we find a relation designated, that of subsumption. We can also say, the object falls under the concept *prime number*. . . . This makes it seem as if the relation of subsumption were a third element supervenient on the object and the concept. This is not the case; rather the unsaturatedness of the concept brings it about that the object, in effecting the saturation, sticks immediately to the concept, without needing any special binding agent (*PPW* 177–178/*NS* 192–193; cf. also *PPW* 193/*NS* 210).

Here, Frege seems to concede, in effect, that the sentence 'The concept *horse* is not a concept' is not true but senseless. He also counts 'falls under' syncategorematic, and thereby does the same for 'the concept F' in those contexts where it has any use at all. From 1914 on, Frege makes similar even more explicit claims that 'the concept *F*' fails to denote anything (*PPW* 238–239, 249–250, 255, 272/*NS* 257–258, 269, 275, 291).

Thus it is possible that Frege held that 'the concept *F*' did not denote anything (as distinguished from merely being too misleading to include in scientific contexts) only after Russell's paradox showed that 'the extension of the concept *F*' was itself defective. Whether or not this is true, 'the concept F' is not said to be substitutable for 'the extension of the concept F' in *Basic Laws.* Passage (A) is, in effect, retracted.

Frege nowhere claims in *Basic Laws* that the two sides of Law (V) express the same sense. This omission may not be significant. He does claim that (V) is a logical law (*BL* 4/*GGA* I, 7) and that it is "what one thinks when one speaks of extensions of concepts."[18] The omission may simply reflect a consciousness that the proposed law was potentially controversial, in that it was admittedly less obvious than the others (*BL* 3/*GGA,* I, 7; *BL* 127/*GGA,* II, 253). In view of Kerry's objections, claims about sameness of sense may have seemed to Frege an unnecessary incitement to doubt. Still, it is striking that in *Basic Laws,* Frege foregoes any serious justification of Law (V), and even expresses doubts about it.

VI. Law (V)—Incomplete Understanding

There is no question that Frege harbored uncertainties about Law (V) in 1893. They surface twice in the opening pages of *Basic Laws* (*BL* 3–4, 25/*GGA* I, vii, xxvi). And in the appendix, reacting to the paradox, he writes, "I have never concealed from myself its lack of self-evidence which the other [Laws] possess, and which must properly be demanded of a law of logic" (*BL* 127/*GGA,* II, 253).

Why was Frege uncertain? It is clear that he was primarily concerned about extracting objects (the extensions) from concepts (*BL* 3–4, 44/*GGA,* I, vii, 14–15; *GGA,* II, 147–148)—a continuation of the doubt expressed in (B) (*FA* 117). He may have continued to worry about the very unKantian notion of particular objects whose existence corresponds to and is justified in terms of no intuition. I believe, however, that intuitive, mathematical worries were more

[18]If the principle had been a law, the two sides of the biconditional would appear to meet a complex criterion for sameness of thought (sense) that Frege proposes in 1906 in a letter to Husserl (*PMC* 70/*WB* 105–106). This criterion is, however, defective in several respects.

basic. Granting the notion of the extension of a concept, one can find certain instances of the principle to be obvious—instances that involve only quantification over individuals, or that use concepts that apply only to individuals. But once one considers quantification over (or concepts that apply to) higher-order objects—extensions of concepts—the principle is less obvious. Of course, this remark benefits from hindsight. It may thus seem anachronistic as an interpretation of Frege's intellectual situation.

I do not think that it is. At the time he proposed Basic Law (V), Frege did lack acquaintance with Russell's paradox; so the specific danger for (V) of allowing indiscriminate quantification over higher-order objects could not have been apparent to him. On the other hand, his notion of the extension of a concept was intuitively unclear. It had previously been given little or no mathematical work. Frege explained the notion in terms of courses of values in 1891 (*G&B* 25, 30–32/*KS* 129, 133–135). And courses of values, especially those explained as objects derived from functions onto truth values, were completely new to the mathematical scene (*BL* 6–7/*GGA*, I, 9–10). Uncertainty about Basic Law (V) must have derived partly from a felt unclarity about the notions of the extension of a concept and the course of values of a function. Frege was explicating the familiar notion of number, which he regarded as insufficiently determinate in its conventional mathematical employment, in terms that were substantially less familiar.[19]

[19]How *should* we understand the notions of the extension of a concept and the course of values of a function, as Frege uses them? I think that there is no clear answer. An illuminating one would require new mathematical work as well as philosophical explication. Frege provides an intuitive means of representing courses of values in terms of geometrical graphs, where the argument of the function is the numerical value of the abscissa and the value is the numerical value of the ordinate. (*G&B* 25/*GGA*, I, 129). No doubt this gives us some hold on the notion. We are then quite ready to explicate this notion of a graph in set-theoretic terms—as a set of ordered couples, where the members of the couples are the arguments and values of the function. This explication is, however, seriously at odds with Frege's own standpoint. The problem is not merely that Frege's notion of the course-of-values of a predicate lacks any limitations on the objects that can form members of the ordered couples. No consistent set theory is free from some such limitations; and one can hardly require a rationalized explication of Frege's notion to match it inconsistency for inconsistency. Rather, the point is that the particular limitations imposed by the dominant, iterative conception of sets are very unFregean. Frege conceived courses-of-values as being projections from predication,

For whatever reason, Frege himself did not find his notion and the associated axiom intuitively compelling. He seems to have struggled to clarify them between 1884 and 1893 with incomplete success. From Frege's own point of view, of course, the concept *extension of a concept* (or the relevant sense) could not itself be unclear (*FA* vii; *KS* 122). In later work, Frege speaks of the expression as being defective and as lacking a *Bedeutung*. But how might he have rationalized the difficulty he was in before 1903?

The difficulty was that Frege found axiom (V) subject to doubt, but maintained that it expressed a logical law. On Frege's own view, logical laws were self-evident and undeniable without plunging thought into "complete confusion" (*BL* 127/*GGA* II, 251; *FA,* section 14). In 1891, and probably since 1884, the difficulty was even sharper: the two sides of the axiom were held to express the same sense—the same thought. Yet Frege harbored sufficiently strong

not constructions from elements (cf. section III). The idea of explicating a course of values, much less a *function,* in terms of ordered couples would have appalled him. From his point of view, a course-of-values' falling in its own extension is no more peculiar than a predicate's applying to itself. By contrast, a set's belonging to itself is ruled out from the outset by the iterative conception. Another way to put the point is that Frege's extensions of concepts violate the axiom of foundation. In fact, if the extensions of concepts are conceived as sets of ordered couples, not a single extension of a concept is well-founded in either the argument or the value position of the ordered couples. From the perspective of the iterative conception of set, this is tantamount, I think, to saying that courses of values are not sets at all.

There are, to be sure, set theories, often called "deviant," that violate the axiom of foundation. But I find it doubtful whether any of these provides an intuitively satisfying reconstruction of Frege's notion of courses of values (although making this point stick would require more argument than I am prepared to give here). The lambda calculus provides some explication of a Frege-like conception of function and concept—and even exhibits some ambivalence over the concept-object distinction. But current model-theoretic interpretations of the lambda calculus do not seem to me to be illuminatingly interpreted as reconstructing Frege's notion of the extension of a concept. Cf. D.S. Scott, "Lambda Calculus: Some Models, Some Philosophy" in *The Kleene Symposium,* Barwise, Keisler, & Kunen, eds. (Amsterdam: North-Holland, 1980); and Albert R. Meyer, "What is a Model of the Lambda Calculus?" (Xerox), Laboratory for Computer Science, MIT, Cambridge, Mass. Perhaps this is just as well; or perhaps the situation will change. But as of now, it seems fair to say that Frege's notion has not found an intuitively attractive, mathematically fruitful counterpart.

doubts about the axiom to have considered radically different ways of founding arithmetic.

One may interpret Frege as lacking a coherent rationalization of the difficulty—as simply torn between the blandishments of the axiom and the warnings of his intuitive conscience. I think, however, that the rationalist elements in his epistemology that we noted in section I provided him with an explication of his situation. In writings early and late, Frege emphasizes that one may not clearly apprehend the concepts (senses) expressed by expressions one uses. Fully grasping the content or sense of an expression—and bringing its conventional significance up to the level of the sense it expresses—depended upon logical analysis. Such analysis involved refining the old language, or introducing new language, and embedding that language in a rigorous logical *theory* in such a way as to elicit thorough understanding of the truth conditions of, and logical relations among, senses expressed by sentences containing the refined language. Thus logical analysis was not separable from the acquisition of logico-mathematical knowledge. Frege thought that one attained insight into the relevant concepts or senses only through developing a theory and seeing it work. This rather pragmatic emphasis on the interdependence of theory and understanding is an integral part of Frege's rationalist conception. Frege retains the traditional rationalist insistence on the close relation between understanding or insight and *a priori* propositional knowledge. But he reverses the traditional order of priority. For him, full understanding depends on, or at most is co-equal with, knowledge, which derives from logico-mathematical theory. Frege's pragmatic emphasis occurs infrequently but repeatedly throughout his writings (cf. *PPW* 33/*NS* 37; *KS* 122–124, 369; *BL* 7, 25/*GGA* I, x, xxvi). Until a theory is developed and logical analysis fully carried out, a full grasp of the refined language may not be achieved.

The expression 'extension of a concept' was just such new or refined language. The attempt to interpret axiom (V) as a biconditional whose two sides expressed the same sense or thought, and indeed the mere postulation of the axiom as expressing a self-evident law of logic, would have been patently untenable *only* on the assumption that the sense of the refined language was perfectly grasped. It could not have surprised Frege that perfect mastery of the notion was difficult to attain (*FA* vii; *KS* 122). He must have hoped that the law's self-evidence would emerge when the senses of the axiom and the expression 'extension of a concept' were fully grasped. (Further evidence for this speculation occurs in the method of *BL,* section 10.)

Frege's rationalist point of view is unquestionably strange from a contemporary perspective. The taproot of the strangeness is, I think, the conception of fully determinate senses or thoughts that are completely independent for their representational properties of human practices or understanding. This conception has as a corollary the conception of self-evident thoughts that no one understands. Few have found these conceptions palatable. I believe that the point of view has some strengths that survive this unpalatability. But here is not the place to explore them.

A common opinion regarding the turn-of-the-century paradoxes is that Law (V), and the closely associated naive comprehension axiom, were originally regarded as obvious—that their defeat at the hands of paradox came as a complete surprise. Surprise there was. But this interpretation underplays the degree of ferment and unclarity over foundational notions in the late nineteenth century. Analogs of the common opinion as applied to other mathematicians of the period need drastic revision. But the view fits even Frege poorly. The evidence indicates that he spent several years trying to dispense with his axiom and, alternatively, to provide it with an attractive exposition and intuitive justification. It is hard not to admire the persistence of his self-questioning.

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