Frege on Apriority*

Frege's logicism incorporated both a set of purported proofs in mathematical logic and an investigation into the epistemology of arithmetic. The epistemological investigation was for him the motivating one. He saw his project as revealing "the springs of knowledge" and the nature of arithmetical justification. Frege maintained a sophisticated version of the Euclidean position that knowledge of the axioms and theorems of logic, geometry, and arithmetic rests on the self-evidence of the axioms, definitions, and rules of inference.[1] The account combines the traditional rationalist view that understanding and what seems obvious are fallible, and that successful understanding is very hard to come by, with his original insistence that understanding depends not primarily on immediate insight but on a web of inferential capacities.

Central to Frege's rationalism is his view that knowledge of logic and mathematics is fundamentally apriori. In fact, near the end of The Foundations of Arithmetic he states that the purpose of the book is to make it probable that "the laws of arithmetic are analytic judgments and consequently apriori".[2] In this essay I want to discuss Frege's conception of apriority, with particular reference to its roots in the conceptions of apriority advanced by Leibniz and Kant.

Frege advertised his notion of apriority as a "clarification" of Kant's notion. It is well-known that Frege did not read Kant with serious historical intent. But even allowing for this fact, his advertisement seems to me interestingly misleading. I believe that his notion is in important respects very different from Kant's and more indebted to Leibniz.

I.

Frege's only extensive explication of his conception of apriority occurs early in The Foundations of Arithmetic. He begins by emphasizing that his conception concerns the ultimate canonical justification associated with a judgment, not the content of truths:

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[1] Burge, (1998a).

[2] Frege (1884), section 87. (Translations are mine. I have consulted Austin's free but often elegant renderings. I will henceforth cite this book by section under the abbreviation "FA" in the text.) Frege's view of analyticity has been more often discussed than his view of apriority. Essentially he takes a proposition to be analytic if it is an axiom of logic or derivable from axioms of logic together with definitions. He rejects conceptions of analyticity that would tie it to containment or to emptiness of substantive content.

> These distinctions between a priori and a posteriori, synthetic and analytic, concern, as I see it,* not the content of the judgment but the justification for the judgment-pronouncement [Urteilsfallung]. Where there is no such justification, the possibility of drawing the distinctions vanishes. An apriori error is thus just as much nonsense as a blue concept. When a proposition is called a posteriori...in my sense, this is not a judgment about the conditions, psychological, physiological, and physical, which have made it possible to form the content of the proposition in our consciousness; nor is it a judgment about the way in which some other man has come, perhaps erroneously, to believe it true; rather, it is a judgment about the ultimate ground upon which rests the justification for holding it to be true.
> *(Frege's footnote): By this I do not, of course, want to assign a new sense but only meet [treffen] what earlier writers, particularly Kant, have meant. (FA, section 3.)

Frege writes here of apriori judgments. But afterwards he writes of apriori propositions and then apriori truths, and eventually (FA, section 87, cf. note 1) apriori laws. These differences are, I think, not deeply significant. Judgments in Frege's sense are idealized abstractions, commitments of logic or other sciences, not the acts of individuals. Individuals can instantiate these judgments through their acts of judgment, but the abstract judgments themselves seem to be independent of individual mental acts. Truths and judgments are, of course, different for Frege. But the difference in Frege's logic concerns only their role in the logical structure. Some truths (true antecedents in conditionals) are not judged. They are not marked by the assertion sign. But everything that is judged is true.[3]

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[3] This doctrine is of a piece with Frege's view that (in logic) inferences can be drawn only from truths. Here he means not that individuals cannot infer things from falsehoods, but rather that the idealized inferences treated in logic proceed only from true axioms. Inferences for Frege are steps in proofs that constitute ideal, correct justifications that exhibit the natural justificatory order of truths. Michael Dummett seems to me to get backwards Frege's motivations for the view that proofs have to start from true premises and that one should not derive a theorem by starting with a supposition. Dummett claims that Frege believed that a complete justification must derive from premises of which no further justification is possible because of his rejection of inference from reductio or from other suppositions. Cf. Dummett, (1991), pp. 25-6. It seems clear that Frege rejected such inference because he thought of proofs as deductive arguments that reveal natural justificatory

Only truths or veridical judgments can be apriori for Frege. He writes that an apriori error is as impossible as a blue concept. Frege justifies his claim that only truths can be apriori by claiming that apriority concerns the nature of the justification for a judgment. Of course, some judgments can be justified without being true. But Frege seems to be focused on justifications--deductive proofs from self-evident propositions--that cannot lead judgment into error. Here Frege signals his concern with canonical, ideal, rational justifications, for which the truth-guaranteeing principles and proofs of mathematics and logic provide the paradigm.

In predicating apriority of truths and judgments, understood as canonical commitments of logic and mathematics, Frege is following Leibniz.[4] Leibniz gave the first modern explication of apriority. He maintained that a truth is apriori if it is knowable independent of experience.[5] Since Leibniz explicitly indicates that one might depend psychologically on sense experience in order to come to know any truth, he means that a truth is apriori if the justificational force involved in the knowledge's justification is independent of experience.

Like Leibniz, Frege conceives of apriority as applying primarily to abstract intentional structures. Leibniz applied the notion not only to truths but to proofs, conceived as abstract sequences of

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order. I do not see that this view is incompatible with allowing "proofs" <u>in the modern sense</u> that proceed without axioms, by natural deduction. Frege's conception of proof is very different from the modern one. It is concerned with an ideal, natural order of justification. Leibniz also thought of reductios as second-class proofs because they do not reveal the fundamental order of justification. Cf. Leibniz, (1705; 1765; 1989), III, iii, 15. Cf. also Adams, (1994), pp. 109-110. I believe that Dummett may be right in holding that Frege's actual mathematical practice may have been hampered by too strict a focus on the justificatory ideal.

[4] Frege makes explicit his dependence on Leibniz on these matters in Frege (1884), section 17: "...we are concerned here not with the mode of discovery but with the ground of proof; or as Leibniz' says, "the question here is not about the history of our discoveries, which differs in different men, but about the connection and natural order of truths, which is always the same"." Frege draws his quotation of Leibniz from Leibniz (1705; 1765; 1989), IV, vii, 9.

[5] Cf. Leibniz, (1705; 1765; 1989), IV, ix, 1, 434; Leibniz (1989), "Primary Truths", p. 31; Leibniz (1989), "On Freedom", p. 97.

truths.[6] Frege assumes that all justifications are proofs, indeed deductive proofs. The apriority of a justification (a series of truths constituting a deductive argument) resides in the character of the premises and rules of inference--in the "deepest ground [justification, Grund] on which a judgment rests". Like Leibniz, Frege thinks that there is a natural order of justification, which consists in a natural justificatory order among truths.

Frege's definitional explication of apriority continues directly from the passage quoted above:

> Thus the question is removed from the sphere of psychology and referred, if the truth concerned is a mathematical one, to the sphere of mathematics. It comes down to finding the proof and following it back to the primitive truths....For a truth to be aposteriori, it must be that its proof will not work out [auskommen] without reference to facts, i.e., to unprovable truths which are not general [ohne Allgemeinheit], and which contain assertions about determinate objects [bestimmte Gegendstande]. If, on the contrary, it is possible to derive the proof purely from general laws, which themselves neither need nor admit of proof, then the truth is apriori. (FA, section 3.)[7]

The notion of a fact about a determinate object in Frege's explication of aposteriori truth is reminiscent of Leibniz' identification of aposteriori truths with truths of "fact", as contrasted with truths of reason.[8] Frege and Leibniz agree in not seeing truths of reason as any less "factual" than truths of fact. The point is not that they are not factual, but that they are not "merely" factual, not merely contingent happenstance. They are

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[6] Cf. Leibniz, (1989) "First Truths", and Leibniz (1714), section 45. There are also occasional attributions of apriority to knowledge or acquaintance. But Leibniz is fairly constant in attributing apriority primarily to truths and proofs.

[7] Austin's translation mistakenly speaks of giving or constructing a proof, which might suggest that the definition concerns what is possible for a human being to do. In fact, Frege's language is abstract and impersonal. His account concerns the nature of the mathematical structures, not human capacities. Austin translates "bestimmte Gegenstande" as "particular objects". In this, I think that he is capturing Frege's intent, but I prefer the more literal translation.

[8] Cf. e.g. Leibniz, (1705; 1765; 1989), IV, vii, 9, 412; IV, ix, 1, 434.

principles that are fundamental or necessary to the very nature of things. The point that apriori truths are general is basic to the Leibniz-Frege conception of apriority. I will return to this point.

Frege departed from Leibniz in thinking that apriori truths include both truths of reason and synthetic apriori truths that involve a combination of reason and geometrical spatial intuition. In this, of course, Frege follows Kant. I believe, however, that Frege's departure from Leibniz on this point is not as fully Kantian as it might first appear. I shall return to this point as well.

Mill had claimed that all justification ultimately rests on induction.[9] Turning Mill virtually on his head, Frege holds that empirical inductive justification is a species of deductive proof, which contains singular statements together with some general principle of induction as premises (FA, section 3). He does not make clear what he considers the form of the deduction to be. And he does not indicate in his definition of aposteriori truths how he thinks singular judgments about "facts" are justified. Presumably he thinks the justification depends in some way on sense experience. It seems likely that he regarded sense-perceptual observations of facts as primitively justified aposteriori. For our purposes, it is enough that Frege thought that justifications relevant to apriori truths are either deductive proofs or self-evident truths. Such justifications have to start with premises that are self-evident and general.

Frege assumed that all apriori truths, other than basic ones, are provable within a comprehensive deductive system. Goedel's incompleteness theorems undermine this assumption. But insofar as one conceives of proof informally as an epistemic ordering among truths, one can perhaps see Frege's vision of an epistemic ordering as worth developing, with appropriate adjustments, despite this problem.[10]

Frege writes that the axioms "neither need nor admit of proof". This phrase is indicative of Frege's view of proof as a canonical justificational ordering of truths, or ideal judgments, that is independent of individual minds or theories. Any truth can be "proved" within some logical theory, in the usual modern sense of the word "prove". But Frege conceived of proof in terms of natural or canonical justification. He saw some truths as fundamental "unprovable" truths, axioms or canonical starting points in a system of ideal canonical justification. Such primitive truths do not need

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[9] J.S. Mill (1843), II,VI,1.

[10] Michael Dummett, (1991), pp. 29-30, in effect makes this point. Dummett errs, however, in thinking that Frege is concerned with what is knowable by us (cf. ibid, pp. 24, 26, 28-9). There is no such parameter in Frege's account. The natural order of justification among truths is conceived as a matter that is independent of whether we can follow it.

proof in that they are self-evident or self-justifying. And they cannot be justified through derivation from other truths, because no other truths are justificationally more basic. Thus they do not admit of proof in his sense. The formula of basic truths' and axioms' neither needing nor admitting of proof can be found verbatim in Leibniz, from whom Frege surely got it.[11]

In introducing his conception of apriority, Frege follows the traditional rationalist practice of indicating the compatibility of apriority with various sorts of dependence on experience. In particular, Frege notes that a truth can be apriori even though being able to think it, and learning that it is true, might each depend on having sense experience of facts.[12] Whether a truth is apriori depends on the nature of its canonical justification. Thus one could need to see symbols or diagrams in order to learn a logical or mathematical truth. One could need sense experience--perhaps in interlocution or simply in observing various stable objects in the world--in order to be able to think with certain logical or mathematical concepts. Perhaps, for example, to count or to use a quantifier, one needs to be able to track physical objects. But these facts about learning or psychological development do not show that the propositions that one thinks, once one has undergone the relevant development, are not apriori. Whether they are apriori depends on the nature of their justification. Frege thinks that such justification in logic and mathematics is independent of how the concepts are acquired, and independent of how individuals come to recognize the truths as true.

In his discussion of Mill's empiricism, Frege reiterates the point:

> If one calls a proposition empirical because we have to have made observations in order to become conscious of its

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[11] Leibniz, (1705, 1765, 1989), e.g. IV, ix, 2; 434. The formula also occurs in Lotze. Perhaps Frege got the phrase from Leibniz through Lotze. Cf. Lotze, (1880); Lotze (1888), section 200. Frege seems, however, to have read Leibniz' New Essays. I discuss this notion of proof and Frege's view of axioms in some detail in Burge (1998a).

[12] Cf. this section of the passage quoted above: "When a proposition is called a posteriori...in my sense, this is not a judgment about the conditions, psychological physiological and physical, which have made it possible to form the content of the proposition in our consciousness; nor is it a judgment about the way in which some other man has come, perhaps erroneously, to believe it true; rather, it is a judgment about the ultimate ground upon which rests the justification for holding it to be true."

> content, one does not use the word "empirical" in the sense in which it is opposed to "apriori". One is then making a psychological statement, which concerns only the content of the proposition; whether the proposition is true is not touched. (<u>FA</u>, section 8).[13]

The key element in the rationalist approach is this distinction between questions about the psychology of acquisition or learning and normative questions regarding the nature of the justification of the propositions or capacities thus learned.

I say "propositions or capacities". Frege follows Leibniz in predicating apriority of propositions, or more particularly, truths, or sequences of truths--<u>not</u> capacities, or mental states, or justifications associated with types of propositional attitudes. Apriority ultimately concerns justification. But Leibniz and Frege share the view that apriority is a feature of an ideal or canonical way of justifying a proposition. For them, a proposition is either apriori or aposteriori, but not both, depending on the nature of the ideal or canonical justification associated with it.[14]

In this, Leibniz and Frege diverge from one distinctive aspect of Kant's thinking about apriority. Like Leibniz and Frege, Kant predicates apriority in a variety of ways--to intuitions, concepts, truths, cognition, constructions, principles, judgments. But whereas Leibniz and Frege predicate apriority primarily of truths (or more

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[13] Frege fixes here on truth, not justification. I think that he is assuming that one learns something about the nature of apriori truths by understanding the proof structure in which they are embedded; and this proof structure constitutes their canonical justification. Cf. Frege (1884) section 105.

Substantially the same distinction between the nature of a truth (and ultimately its justification) and the ways we come to understand the relevant proposition or to realize its truth is made by Leibniz, (1705, 1765, 1989), Preface, pp. 48-9; IV, vii, 9; and by Kant (1781, 1787), A1, B1.

[14] Some modern philosophers who take apriority to be predicated primarily of propositions call a proposition apriori if it <u>can</u> be justified apriori. Apriori justification is then explained in some non-circular way. Cf. Kripke (1972), p. 34. This formulation avoids commitment to that way's being canonical or ideal. But it also leaves out a serious commitment of such rationalists as Leibniz and Frege. For them, apriori justification is the best and most fundamental sort of justification. When something can be known or justified apriori, that is the canonical way.

fundamentally, proofs of truths), Kant predicates apriority primarily of cognition and the employment of representations. For him apriori cognition is cognition that is justificationally independent of sense-experience, and of "all impressions of the senses".[15] Apriori cognition is for Kant cognition whose justificational resources derive purely from the function of cognitive capacities in contributing to cognition. Apriori employment of concepts (or other representations) is employment that carries a warrant that is independent of sensory experiences. Aposteriori cognition is cognition which is justificationally derivative, in part, from sense experiences.

Both conceptions are ultimately epistemic. Frege very clearly states that his classification concerns "the ultimate ground on which the justification for taking [a truth] to be true depends" (FA, section 3). Both sharply distinguish epistemic questions from questions of actual human psychology. Both take apriority to hinge on the source or method of warrant.

One might think that the main difference lies in the fact that Kant acknowledges more types of warrant as sources of apriority. Leibniz and Frege allow self-evidence and proof. Kant allows, in addition, constructions that rest on pure intuition and reflection on the nature of cognitive faculties.

I think, however, that this difference is associated with a fundamentally different orientation toward apriority. Frege and Leibniz explicate the nature of apriority in terms of a deduction from general basic self-evident truths. All that matters to apriority is encoded in the eternal, agent-independent truths themselves. For deductive proof turns entirely on such contents. An individual's being apriori justified consists just in thinking through the deductive sequence with understanding.

For Kant, the apriority of mathematics depends on possible constructions involving a faculty, pure intuition, that does not directly contribute components of truths (the conceptual components of propositions or thoughts). According to Kant, the proofs in arithmetic and geometry are not purely sequences of propositions. The justifications, both in believing axioms and in drawing inferences from them, must lean on imaginative constructions in pure intuition, which cannot be reduced to a sequence of truths. The intuitive faculty contributes singular images in apriori imagination. Not only are these not part of an eternal order of conceptual contents. The proofs themselves essentially involve mental activity and make essential reference, through intuition, to particulars. For Kant these particulars are aspects of the mind. So the structure of a mathematical proof makes essential reference to possible mental particulars. It is not an eternal sequence of truths that are fundamentally independent of particulars.

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[15] Kant, (1781, 1787), B2-3.

Kant's conception of synthetic apriori cognition thus depends on an activity, a type of synthesis involved in the making of intuitive constructions in pure imagination. It is significant that, unlike Leibniz and Frege, he makes no appeal to self-evidence. That is, he does not claim that the evidence for believing the basic truths of geometry and arithmetic is encoded in the truths. In arithmetic he does not even think that axiomatic proof is the basis of arithmetical practice.[16]

This orientation helps explain Kant's tendency to predicate apriority of cognition rather than truths. It is also at the root of his concentration, in his investigation of apriori warrant, on the functions and operations of cognitive capacities, not on the nature of conceptual content and the relations among truths. The orientation makes the question of what it is to <u>have</u> a justification much more complex and interesting than it is on the Leibniz-Frege conception. And it ties that question more closely to what an apriori warrant is.

Kant's shift in his understanding of apriority from the content of truth and of proof-sequences of propositions to the character of cognitive procedures opens considerably more possibilities for understanding sources of apriority, and for seeing its nature in capacities and their functions, or even in specific acts or mental occurrences, rather than purely in propositional forms. Kant's account does not depend on empirical psychology, but it does center on a transcendental psychology of the cognitive capacities of any rational agent.[17]

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[16] Kant (1781, 1787), A164/B205.

[17] The relation between the two approaches is complex and needs further exploration. But it is worth remarking that Kant's approach has this advantage of flexibility: For Leibniz and Frege, a truth is either apriori or aposteriori. It is apriori if its canonical or ideal mode of justification is apriori. Its canonical mode of justification is apriori if it is situated in a natural proof structure either as a primitive truth--which does "not need or admit of proof"--or as a deductive consequence from primitive truths and rules of inference. On Kant's conception, a truth can be known or justifiably believed either apriori or aposteriori, depending on what form of justificational procedure supports it for the individual. For on this conception, apriority is predicated not primarily of truths but of modes of justification, or even states of cognition. Kant did not make use of this flexibility. Its possibility is, however, implicit in his conception.

Michael Dummett, (1991), p. 27, writes "it is natural to take Frege as meaning that an a priori proposition may be known a posteriori: otherwise the status of the proposition would be determined by any <u>correct</u> justification that could be given for it."

A second way in which Frege diverges from Kant is that his explanation of apriority in The Foundations of Arithmetic makes no mention of sense experience. Instead he characterizes it in terms of the generality of the premises of its proof.[18] Both Leibniz and Kant characterize apriority directly in terms of justificational independence of experience. Unlike Leibniz, Kant consistently takes experience to be sense experience. Since any modern notion of apriority seems necessarily tied somehow with justificational independence of experience, Frege's omission is, strictly speaking, a

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He goes on to discuss whether there are any propositions that can be known only apriori. I have no quarrel with Dummett's substantive discussion. But his historical reasoning is off the mark. Frege's characterization takes apriority to apply to truths or idealized judgments. There is no relativization to particular ways of knowing those truths. A truth or judgment-type is either apriori or not. A truth or judgment is apriori if its best or canonical justification proceeds as a deductive proof from general principles that neither need nor admit of proof. Dummett fails to notice that there is no clear meaning within Frege's terminology for a question whether a truth can be known both apriori and aposteriori. That question can be better investigated by shifting to a Kantian conception of apriority. Dummett slides between the two conceptions. Frege could certainly have understood and accepted the Kantian conception; but he did not use it or propose it.

Dummett's reasoning to his interpretation is unsound. Suppose for the sake of argument that we reject the view that an apriori proposition can be known aposteriori. (I myself would resist such a rejection.) We might allow that there are empirical justifications for something weaker than knowledge for all propositions. For example, we might strictly maintain the Leibniz-Frege conception and insist that apriori truths can be known only apriori. Then it simply does not follow that the status of the proposition would be determined by any correct justification that could be given for it. The status would still be determined by the best justification that could be given for it. Oddly, Dummett clearly sees that this is Frege's conception elsewhere--Dummett (1991), p. 23.

[18] As Dummett notes, Frege's definition of "apriori" is cast in such a way that the premises of apriori proofs are counted neither apriori nor aposteriori. Dummett (1991), p. 24. I think that Dummett is correct in thinking this an oversight of no great significance. It would be easy and appropriate to count the primitive truths and rules of inference apriori.

mischaracterization of the notion of apriority.[19]

From one point of view, this omission is not of great importance. Frege evidently took his notion of apriority to be equivalent with justificational independence of sense experience. His discussion of Millian empiricism follows his definition of apriority by a few pages. In those sections he repeatedly writes of "observed facts", apparently picking up on the notion of fact that appears in his definition of aposteriority (FA, sections 7-9). He seems to assume that mere "facts"--unprovable truths that are not general--can enter into justifications only through observation.[20] So a proof's depending on particular facts would make it rest on sense-experience. Moreover, his criticism of Mill explicitly takes "empirical" to be opposed to "apriori" (FA, section 8).

In Foundations of Arithmetic section 11, Frege infers from a proof's not depending on examples to its independence of "evidence of the senses". The inference suggests that he thought that a proof from general truths necessarily is justificationally independent of sense experience. At the beginning of The Basic Laws of Arithmetic he states the purpose of The Foundations of Arithmetic as having been to make it plausible that arithmetic is a branch of logic and "need not borrow any ground of proof whatever from either experience or intuition."[21] Here also Frege assumes that a proof's proceeding from general logical principles entails its justificational independence from experience or intuition. Frege commonly accepts the Kantian association of intuition (in humans) with sensibility, so here again it is plausible that he meant by "experience" "sense experience".

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[19] There are differences between Leibniz' and Kant's accounts on this point that are relevant, but which I intend to discuss elsewhere. Leibniz often characterizes apriority in terms of justificational independence of experience. Leibniz sometimes allows intellectual apprehension of intellectual events to count as "experience". Kant firmly characterizes apriority in terms of justificational independence of _sense_ experience. Kant's specification has important consequences, and makes his view in this respect the more modern one. It was taken up by Mill, the positivists, and most other twentieth century empiricists. For purposes of epistemological discussion, "experience" has come to mean _sense experience_.

[20] Precisely the same inference can be found in Leibniz (1705; 1765; 1989), Preface 49-50.

[21] Frege (1893, 1902), section 0. Compare this characterization of the earlier book's purpose with the one quoted from Frege (1884), section 87 (cf. note 1 above). It is possible that the latter characterization constitutes a correction of the mischaracterization of apriority in Frege (1884), section 3.

In very late work, forty years after the statement of his definition, Frege divides sources of knowledge into three categories: sense perception, the logical source of knowledge, and the geometrical source of knowledge. He infers in this passage from a source's not being that of sense perception that it is apriori.[22]

So Frege took his definition of apriority in terms of derivation from general truths to be equivalent to a more normal definition that would characterize apriority in terms of justificational independence from sense experience. Still, the non-standardness (incorrectness) of Frege's definition is interesting on at least two counts. First, its focus on generality rather than independence from sense-experience reveals ways in which Frege is following out Leibnizean themes but in a distinctively Fregean form. Second, the definition is backed by a presupposition, shared with Leibniz, that there is a necessary equivalence between justifications that start from general principles and justifications that are justificationally independent of sense experience. It is of some interest, I think, to raise questions about this presupposition.

II.

Let us start with the first point of interest. Leibniz and Frege both see apriori truth as fundamentally general. Apriori truths are derivable from general, universally quantified, truths. Both, as we have seen, contrast apriori truths with mere truths of fact. Leibniz held that mere truths of fact are contingent, and that apriori truths are necessary. He took necessary truths to be either general or derivable from general logical principles together with definitional analyses and logical rules of inference. So for Leibniz the apriori/aposteriori distinction lines up with the necessary/contingent distinction, and both are closely associated with Leibniz' conception of a distinction between general truths and particular truths.[23]

It is tempting to regard Frege in the same light. As we have seen, Frege even defines apriority in terms of derivability from general truths and aposteriority in terms of derivability from particular truths. But there is little evidence that Frege associated apriority or generality with necessity. In fact, modal categories are strikingly absent from Frege's discussion.

We can gain a more refined understanding of Frege's differences from both Leibniz and Kant by contrasting his terminology with Kant's. Kant's conception of apriority, as we have seen, is explicitly defined

[22] Frege (1979), pp. 267ff., 276-277; Frege (1983), pp. 286ff., 296-7.

[23] Leibniz, (1705; 1765; 1989), Preface 49-50; IV, vii, 10, 412-3; IV, xi, 13, 445-446.

in terms of a cognition's independence for its transcendental genesis and its justification from sense experience. But he cites two other properties as marks (Merkmale) or sure indications (sichere Kennzeichen) of apriority. One is necessity. The other is strict generality (or universality) (strenge Allgemeinheit).[24]

There are two points to be noted about these remarks. One is that Kant provides these marks or indications not as elements in the definition of apriority, but as signs, which according to his theory are necessarily associated with apriority. In fact, in providing these signs, he takes them to be sufficient for apriority. He does not, in these famous passages, claim that they are necessary conditions.[25] The reason why on his view apriori judgments are associated with necessity and strict generality is not that these associations follow from his definition or conception of apriority. The associations derive from further commitments in Kant's system.

Kant explains strict generality itself in terms of modality. Kant contrasts strict generality with comparative or assumed

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[24] Kant (1781, 1787), B3-4; cf. A2; A91-2/B124. The same point is made in Kant (1790), section 7 (Akademie Ausgabe V, 213). There Kant calls comparative generality "only general" (nur generale), and strict generality "universal" (universale). Compare Leibniz, (1705; 1765; 1989), IV, ix, 14, 446: "The distinction you draw [between particular and general propositions] appears to amount to mine, between "propositions of fact" and "propositions of reason". Propositions of fact can also become general, in a way; but that is by induction or observation, so that what we have is only a multitude of similar facts...This is not perfect generality, since we cannot see its necessity. General propositions of reason are necessary..."

[25] I think that Kant believed that necessity was (necessarily) necessary as well as sufficient for the apriority of a judgment. He clearly believed that being, or being derivable from, a strictly general proposition is sufficient for the apriority of a judgment. Kant surely believed that all apriori judgments are true without any possible exceptions. Whether he believed that all apriori judgments have to be derivable from judgments that are in the form of universal generalizations is more doubtful. I shall discuss this matter below. Whether strict generality was only a sufficient condition (a mark) of apriority, not a necessary one--or whether it was both necessary and sufficient, but understood in such a way as not to entail the logical form of a generalization--is a complex question that I shall leave open. What is certain is that Kant's views on the relation between apriority and both necessity and strict generality depend not merely on his definition or conception of apriority, but on other elements in his system. I believe that rejecting Kant's positions on these relations is compatible with maintaining his conception of apriority.

generality. Comparative generality holds only as far as we have observed.[26] A judgment thought in strict generality "permits no possible exception". Kant infers from this that such a thought is taken as holding absolutely apriori.

Neither Kant nor Leibniz gives any hint of defining apriority in terms of generality. Both appeal, however, to generality in their elucidations of apriority. Frege's use of generality (Allgemeinheit) in his definition is surely inherited from them. Like them he believed that apriority is deeply connected with some form of generality of application, or universal validity. But he interpreted and used his notion of generality differently. He departs from both Leibniz and Kant in defining apriority in terms of generality. He departs from both in saying little about the relation between apriority and necessity. Indeed, his conception of generality differs from both in that he does not connect it to modal notions, seen as independent notions, at all.

Frege does comment on the relation between generality and necessity very briefly in Begriffsschrift. He associates generality with the logical form of the contents of judgments. He claims that apodictic judgments differentiate themselves from merely assertoric ones in that they suggest the existence of general judgments from which the proposition can be inferred. He then writes,

> When I designate a proposition as necessary, I thereby give a hint about the grounds of my judgment. But since the conceptual content of the judgment are not thereby touched, the form of the apodictic judgment has no significance for us.[27]

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[26] Cf. note 24. Strictly speaking comparative generality and strict generality do not seem to be exhaustive categories. It would appear that there are propositions that are comparatively general but which are not true accidental generalizations (there is a counterinstance that simply has not been found); yet true accidental generalizations are not necessary truths. This is because it is possible for there to be true accidental generalizations which have no counter-instances yet observed. (I leave open whether there are also empirical laws which are general but which are not strictly general, in Kant's sense.) It is possible, of course, that Kant means the "we" in "what we have so far observed" in a loose and highly idealized sense. It is conceivable that he intended comparative generality to include all possible actual observations by "us". Given his idealism, he would take this as equivalent to the empirical truth of the generalization. This is a matter that could bear more investigation.

[27] Frege (1879), section 4. The issue is discussed briefly by Gabriel (1996).

Frege seems to think that necessity is not represented in logical form, but is to be explained in terms of a pragmatic suggestion regarding the epistemic grounds for a judgment. Generality for Frege (in the sense relevant to this context) is simply universal quantification. What makes a truth apriori is that its ultimate grounds are universally quantified. So Frege seems to explicate necessity in terms of apriority. Apriority is the notion that Frege attaches in <u>Foundations of Arithmetic</u> to the condition he envisages here in the <u>Begriffsschrift</u> of a judgment's having its ground in general propositions. If anything, Frege explains necessity in terms of the (ordinary) generality of the grounds of the proposition. This contrasts with Kant's explaining (strict) generality in terms of necessity.

I think that Frege was trying to get the effect of the difference between accidental generalizations and empirical laws, on one hand, and necessary generalizations, on the other, while avoiding explicit introduction of independent modal notions. His notion of generality is the simple one of universal quantification. Not just any general truth is apriori, however. Only general truths that are self-evident axioms, or first-truths, or which are derivable from self-evident axioms, or first-truths, are apriori. Apriori generalizations are generalizations whose <u>ultimate justification</u> does not rest on particular truths.

Frege does use the notion of law in his characterization of apriority: "If...it is possible to derive the proof purely from general laws, which themselves neither need nor admit of proof, then the truth is apriori" (FA, section 3).[28] Empirical laws need and admit of "proof", in that they need justification from statements of observation about particulars. It is common to hold that the notion of law contains or implies modal notions. That may well be. But I believe that Frege thought of laws in terms of basic principles in a system of scientific propositions--either an empirical science or a deductive science--not (at least not officially) in terms of any modal or counterfactual element. Empirical laws are basic principles of (idealized, true) empirical scientific systems of true, grounded propositions. But they are not basic in the order of justification: singular observational statements (along with an apriori principle of induction) are supposed to be justificationally prior. Apriori laws differ in just this respect.

So the key idea in distinguishing empirical laws and accidental generalizations from apriori truths is taking apriority to be justificational derivation from general truths, which themselves are self-evident and do not need or admit of proof. Frege's notion of

[28] I believe that Frege's use of "possible" in this remark is dispensable. It is possible to derive the proof in his sense if and only if there is a proof.

generality is fundamentally less modal than Kant's notion of strict generality or universality. It is simply that of universal quantification, where quantification is understood to be unqualifiedly general--to range over everything. Apriority is understood in terms of the priority of generality in justification.[29]

I have no doubt that Frege worked with an intuitive notion of logical validity. This enters his formulation of rules of inference. But the universal validity of logical laws is supposed to lie in their applicability to everything--which includes mathematical and geometrical objects and functions. The mathematical objects provide a sufficiently large and strict subject matter to enable true quantifications in logic and mathematics to have some of the force and effect of necessary truths that purport to quantify over possible objects or possible worlds. This force and effect seems to suffice for Frege's purposes. Frege seems to avoid invocation of an independent notion of modality and of merely possible objects, in epistemology, metaphysics, and logic.

Leibniz took all truths to be deducible in principle from truths of logic. On his view, it is a mere weakness of the finite human intellect that requires it to invoke empirical experience to arrive at ordinary truths about the physical world. Frege joined the rest of mankind in regarding Leibniz' view as overblown (FA, section 15). Of course, he agreed with Leibniz in holding that arithmetic is derivable from logic. Logic is naturally seen as a canon of general principles associated with valid inference. Here Frege sided with Leibniz against Kant in holding that one can derive truths about particular, determinate objects--the numbers--from purely general logical principles. Frege specifically states his opposition to Kant's view that without sensibility, no object would be given to us (FA, section 89). He argues that he can derive the existence of numbers from purely general logical laws. In this, of course, he failed. But the Leibnizean idea of obtaining truths about particular determinate objects from general, logical, apriori principles is fundamental to his logicist project.

It seems to me likely that Frege's opposition to iterative set theory partly derives from the same philosophical picture.[30]

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[29] Frege has another concept of "generality", of course, by which he distinguishes arithmetic and logic, which are completely general in their domain of applicability, from geometry, which applies only to space.

[30] Frege (1884), sections 46-54; Frege (1893; 1902a) 30; Frege (1893; 1902b) I, 2-3; Frege (1984), pp. 114, 209, 228; Frege (1967), p. 104-5, 209-210. The latter passage especially seems to find the problem in the assumption of single things at the base of set theory. The idea that concepts are general and objects must be derivative from

Iterative set theory naturally takes objects, the ur-elements which are the members of sets, as primitive. They may be numbers or unspecified ur-elements, but they are naturally taken as given. Frege thought that an apriori discipline has to start from general principles. And it would be natural for him to ask where the ur-elements of set theory come from. If they were empirical objects, they would not be given apriori. He regarded the null set as an indefensible entity from the point of view of iterative set theory. It collects nothing. He thought a null entity (a null extension) is derivable only as the extension of an empty concept. If one took the numbers as primitive, one would not only be giving up logicism. One would be assuming particular objects without deriving their existence and character from general principles--thus controverting Frege's view of the nature of an apriori subject. If one could derive the existence of numbers from logical concepts, one would not need set theory to explain number theory or, Frege thought, for any other good purpose. Thus it would have been natural for him to see set theory as raising an epistemic puzzle about how its existence claims could be apriori, inasmuch as they appear to take statements about particulars as primitive or given.

Leibniz actually _characterizes_ reason as the faculty for apprehending apriori, necessary truths. These include for him all mathematical truths. As I have noted, Leibniz regards all necessary truths as ultimately instances of, or derivative from, _general_ logical principles together with definitional analyses and logical rules of inference.[31] Generality for Leibniz is a hallmark of reason. As noted, Frege agrees that arithmetic is thus derivative from general logical principles. He takes arithmetic to be an expression of pure reason, and its objects given directly to reason through logical principles (_FA_, section 105).

Kant famously separates apriority and necessity from pure reason in the sense that he holds that some apriori, necessary truths, the synthetic ones, can be known only by supplementing reason with the products of a non-rational faculty for producing singular representations--intuition. For Kant intuition is essentially a faculty for producing _singular_ representations. It is part of his view that synthetic cognition of objects, including synthetic apriori cognition in arithmetic and geometry, must partly rest its justification on the deliverances of intuition. Hence the justification must rest partly on singular representations, and perhaps propositions or thoughts in singular form as well.

Of course, Frege disagrees with Kant about arithmetic. He holds

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principles governing concepts guided his opposition.

[31] Leibniz (1705; 1765; 1989), Preface, 49-50; IV, vii, 10, 412-3; IV, xi, 13, 445-446.

that arithmetic is not synthetic, but analytic--at least in the sense that it is derivative from general logical principles without any need to appeal to intuition. But Frege purports to agree with Kant about geometry (FA, section 89). He agrees that it is synthetic apriori. It is synthetic in that it is not derivable from logic. The logical coherence of non-Euclidean geometries seemed to confirm its synthetic character. Frege also purports to agree that geometry rests on pure apriori intuition.[32] He agrees with Kant in counting intuition a faculty different from the faculty of thought (e.g. FA 26, 90). Frege's agreement with Kant that apriori truths of geometry rest on intuition, a faculty for producing singular representations, puts some pressure on Frege's view that apriori truths must rest on fundamentally general laws. As we shall see, there is some reason to think that Frege's relation to Kant on this matter is not as straightforwardly one of agreement as he represents it to be.

III.

Let us now consider the second point of interest in Frege's characterization of apriority. This is his presumption that his characterization of apriority in terms of the primacy of generalizations in proof is equivalent with the usual post-Kantian characterization in terms of justificational independence from sense experience.

There are at least three areas where both the general characterization and Frege's assumption of equivalence can be challenged. One has to do with certain types of self-knowledge, and perhaps more broadly, certain context-dependent truths. One has to do with geometry. One has to do with arithmetic. I will not go into these issues in depth. But I hope that broaching them will be of both historical and substantive value.

Frege exhibits no interest in *cogito* judgments: judgments like the judgment that I am now thinking. But his characterization of apriority immediately rules them aposteriori, in view of the singularity of their form and their underivability from general laws. Now the question of whether *cogito* judgments are in fact aposteriori is a complex one.

Leibniz is in accord with Frege in counting them aposteriori. He counts them primitive, self-evident truths which nevertheless depend

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[32] Unlike Kant, Frege gives no clear evidence of believing that all synthetic apriori principles rest on intuition. He holds that the principle underlying (non-mathematical) induction is synthetic apriori, but he gives no reason to think that it rests on intuition. This point is made by Michael Dummett (1982), p. 240.

on "experience".[33] What Leibniz means by "experience" is not very clear. His view suffers by comparison to Kant's in its vastly less developed conception of cognitive faculties and of the nature of experience. Sometimes Leibniz associates experience with sense experience. But it appears that he sometimes uses a very broad conception of experience that would include any direct awareness of an object or event, whether or not this awareness proceeds through one of the senses. Thus "experience" for Leibniz, at least at times, seems to include not only what we would count sense experience but intellectual "experience" as well. A conception of apriority as independence from experience in this broad sense would be defensible. Its counting instances of the _cogito_ "aposteriori" would also be defensible.

Frege consistently associates experience with _sense experience_. If he were to relax this association, it would be open to him to side with Leibniz (or one side of Leibniz) here against Kant in counting non-sensory intellectual awareness of particular intellectual events as experience.[34] Such a conception would, however, sever the connection between apriority and independence of the experience of the senses. Frege seems to accept this connection. It has dominated conceptions of apriority since Kant. What seems to me thoroughly doubtful is that our cognition of instances of the _cogito_--and perhaps other indexical thoughts such as _I am here now_ or _I exist_--is justificationally dependent on sense experience. Such cognition seems to depend only on intellectual understanding of the thought content in an instance of thinking it. Contingent, singular truths seem to be apriori in the sense that our warrant to accept them is justificationally independent of sense experience.

If these points are sound, they raise interesting questions about the relation between apriority, reason, and generality. It seems to me natural--at least as a working conjecture--still to regard reason (with Leibniz and Kant) as essentially involved in supplying general principles and rules of inference. A warrant can, however, be justificationally independent of sense experience if it gains its force from either reason or understanding. And understanding essentially

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[33] Leibniz (1705; 1765; 1989), IV, vii, 7, p. 411; IV, ix, 3, p. 434; cf. IV, ii, 1, p. 367.

[34] Kant also thought that instances of the _cogito_ produce no "apriori" cognition. But this view cannot be directly derived from his characterization of apriority alone, as it can be from Leibniz' characterization. Rather Kant's view depends on his very complex (and I think mistaken) theory of the dependence of cognition of one's thoughts in time on inner sense, which ultimately depends, albeit indirectly, for its justificational force on outer sense. I shall not discuss this Kantian view here.

involves singular elements. The view is fundamentally Kantian: Reason is essentially general. Understanding, because of its interdependence with non-rational capacities, is sometimes understanding of truths in singular form that cannot be proved from general truths. Warrant can be apriori if it derives from reason or from understanding, if it does not depend on sense experience for any of the force of its epistemic warrant.

I believe that Kant was mistaken, however, in holding that understanding can yield non-logical cognition only if it applies to the form or deliverances of sensory capacities (and non-logical *apriori* cognition only if it applies to their form). I believe that understanding is capable of yielding non-empirical and non-sensible cognition of thoughts in singular form that are not derivable from general ones. One can, for example, know by intellection and understanding alone that certain of one's intellectual mental events are occurring (or have occurred), or that one is thinking. No invocation of sensible intuition or the form of one's sensory capacities is needed for the the justification that underwrites the relevant knowledge. It seems to me plausible that our understanding sometimes applies to intentional contents that are tokens, instances of indexicals, in singular form.[35]

Perhaps to account for the apriority of our warrant for believing such instances, the warrant must be seen as deriving *partly or in some way* from something general. For example, to understand the self-evidence of an instance of *I am now thinking*, one must understand *I* according to the general rule that it applies to whomever is the author of the thought that contains its instantiation. One must understand a similar general rule for *now*. Thinking according to such rules, one can realize that any instance of *I am now thinking* will be true. This is an entirely general insight. It seems to me plausible to consider a logic for the *forms* of such indexicals as an expression of reason. Here the generality of reason would not reside in the form of the propositional content (which is singular), but in the generality of the rules governing its application. The semantical rule is in general form.[36]

But the realization of the truth of an *instance* of the *cogito* cannot be derived purely from these generalities. It cannot be derived purely from a logic of indexicals or from anything purely general. It must involve an awareness in understanding of an actual event of thinking and a recognition of its content. Thus the warrant cannot rest purely on an inference from general principles. There is

[35] Cf. Burge (1996); Burge (1998b).

[36] For an example of a logic of such singular indexicals, see Kaplan (1989).

something irreducibly singular in the application of the understanding. The warrant depends essentially for its force on the exercise of this singular application. Although the truth--the instance of the cogito--would count as aposteriori on Leibniz' conception and on Frege's conception, it is plausibly apriori on the Kantian conception: The warrant for an instance of thinking it is justificationally independent of sense experience. The warrant depends for its force purely on intellectual understanding applied to a singular instance of a cogito thought. Cf. note 34.

IV.

I turn now to Frege's application of his characterization of apriority to geometry. Frege accepted Kant's doctrine that Euclidean geometry is synthetic apriori. Frege meant by "synthetic" here not derivable from logic. Frege also maintains with Kant that geometry rests on sensible geometrical spatial intuition. With Kant, Frege held the now discredited view that Euclidean geometry is both apriori and apriori-applicable to physical space. It is now tenable to hold that Euclidean geometry is apriori only if one considers it a pure mathematical discipline whose proper application, or applicability, to physical space is a separate and empirical question. I want, however, to discuss the issue of the epistemic status of Euclidean geometry from Frege's perspective.

What did Frege mean by his agreement with Kant about the epistemology of Euclidean geometry? There is no firm evidence that Frege accepted Kant's idealist conception of physical space. Frege's whole philosophy, especially in his mature period, seems out of sympathy with the explanation of apriority in terms of the mind's imposing its structure on the physical or mathematical worlds.[37] Frege articulated his agreement with Kant by agreeing that geometry is based on, or has its "ground" in, pure intuition (FA, sections 12, 89).[38]

For Kant, pure intuition is both a faculty and one product of the faculty. Intuition is a faculty for singular, immediate representations. It represents singular elements of (or in) space or time without being mediated by any further representations that apply to the same semantical values or referents. Pure intuition is the faculty itself, considered independent of any passively received, sensational content. For Kant intuition could be either an

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[37] For an elaboration of some aspects of this theme, see Burge (1998a).

[38] Cf. also "On a Geometrical Representation of Imaginary Forms in the Plane" in Frege (1984), p. 1; or Frege (1967), p. 1; "Methods of Calculation based on an Extension of the Concept of Quantity" in Frege (1984), p. 56; or Frege (1967), p. 50.

intellectual faculty (in which case its exercises would always be pure), or a sensible one.[39] We humans have, according to Kant, only sensible intuition. Pure sensible intuition is the structure of the faculty which is constant regardless of what sensational contents one receives in sense-perceptual experience or produces in empirical imagination.

Kant also believed that pure sensible intuition could itself yield pure representations as product--pure formal intuitions.[40] Such representations are representations of elements in the structure of space and time. Given his idealism, these elements were supposed to be features of the structure of the faculty of sensible intuition. Intuitions of all sorts are characterized by Kant as being objective representations that are both immediate and singular.[41]

If one strips this view of its idealist elements, one can regard pure sensible intuition as a faculty for intuiting the pure structure (not of the faculty itself but) of mind-independent space and time. Frege shows no interest in pure temporal intuition. Of course, in his mature period he rejects Kant's view that arithmetic rests on pure temporal intuition, or intuition of any sort. He believed, however, that we have a capacity for pure spatial intuition. He believed that Euclidean geometry is in some way grounded in exercises of this capacity. Like Kant, Frege associates the capacity for pure intuition (in humans at least) with sensibility--the capacity for having sense experiences. He distinguishes it from a capacity for conceptual thought (FA, section 14).

What interests me is Frege's understanding of the singularity of

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[39] Frege shows a certain superficiality in his reading of Kant in Frege (1884), section 12. There he first notes that in his Logic Kant defines an intuition as a singular representation, noting that there is no mention there of any connection with sensibility. He further notes that in the Transcendental Aesthetic part of Critique of Pure Reason the connection is added (hinzugedacht), and must be added to serve as a principle of our cognition of synthetic apriori judgments. He concludes that the sense of the word "intuition" is wider in the Logic than in the Critique. But it is not wider. In both books intuition is characterized in terms of singularity (and in the Critique sometimes in terms of immediacy as well). Cf. Kant (1800) section I.1; Kant (1781; 1787), A320/B376-7. Kant intentionally leaves sensibility out of the characterization of the notion in both books because he takes intellectual (non-sensible) intuition to be one possible type of intuition--possible in principle, though not for humans.

[40] Kant (1781, 1787) B160.

[41] Kant (1781, 1787) A320/B377.

pure intuition and its relation to his characterization of apriori truths as following from general principles that do not need or admit of proof. He cites and does not reject Kant's conception of intuitions as individual representations (FA, section 12). He regards the axioms and theorems of Euclidean geometry as apriori. So he thought that they are, or follow from, general principles that do not need or admit of proof. The proof must work out without reference to unprovable truths which are not general and which contain assertions about determinate objects [bestimmte Gegendstaende]. Kant takes intuitions to play a role in the warrant of some geometrical axioms and rules of inference. What is the epistemic role in Frege's view of pure intuitions--which for Kant are certainly singular, not general--in warranting the axioms of geometry?

Frege is aware of this question. He speaks to it in section 13 of The Foundations of Arithmetic. He writes,

> One geometrical point, considered in itself, is not to be distinguished any way from any other; the same applies to lines and planes. Only if more points, lines, planes are comprehended at the same time in an intuition, does one distinguish them. From this it is explicable that in geometry general propositions are derived from intuition: the intuited points, lines, planes are really not particular (besondern) at all, and thus they can count as representatives of the whole of their kind. But with numbers it is different: each has its own particularity (Eigentumlichkeit).[42]

Frege does not use language in this passage that connects precisely with the language of his characterization of apriority.[43] Perhaps he simply believed that since the relevant objects of intuitions are not "particular" (besondern), they are not "determinate objects" (bestimmte Gengenstaende). (Cf. the definition of aposteriority.) Or perhaps he believed that pure intuition's

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[42] Frege does not make it clear why it matters that one can distinguish the objects of intuition from one another only if they are comprehended in a complex intuition, or why this fact shows that the objects are not really particular at all.

[43] In a paper on Hilbert, Frege seems to sympathize with the idea that axioms assert basic facts about intuition. But he is focused on Hilbert's view that axioms both assert and define things. Frege's main point is that axioms cannot do both; he clearly believes that they assert something. There is little in the passage to help us with his attitudes toward the singularity of intuitions or their precise role in the epistemology of geometry. Cf. Frege (1984) pp. 275-7; Frege (1967) pp. 264-6.

contribution to the justification of general truths lies not in its representation of determinate objects (the individual lines and planes that it represents), but of aspects of them that are not particular to those objects. He may have thought that although we must be presented with particulars in pure intuition, the warranting power of the intuition lies only in geometrical properties that are invariant under Euclidean transformations. In either case, Frege does not give a precise explanation of how intuition helps "ground" (FA section 12) our knowledge. Hence Frege gives no precise explanation of how his view of the apriority of geometry is compatible with his view of its depending on pure intuition--a faculty for singular representation.

Nevertheless, the main thrust of the passage seems to be to downgrade the role of the particularity of the geometrical objects, and of the singularity of thoughts about them, in the "derivation" of general truths. In fact, Frege says that the objects of pure intuition in geometrical imagination are not genuinely particular. He seems to see the lines that he regards as objects of intuition as types. So they can serve as representatives whose characteristics that are shareable with relevantly similar objects are all that matter for arriving at general propositions. It is difficult to see here how Frege's view relates to Kant's, even bracketing the fact that Frege does not advocate Kantian idealism.

Let us approach this question by first comparing the just quoted passage from Frege with a passage in Leibniz. Leibniz writes:

> But I do not agree with what seems to be your view, that this kind of general certainty is provided in mathematics by "particular demonstrations" concerning the diagram that has been drawn. You must understand that geometers do not derive their proofs from diagrams, although the expository approach makes it seem so. The cogency of the demonstration is independent of the diagram, whose only role is to make it easier to understand what is meant and to fix one's attention. It is universal propositions, i.e. definitions and axioms and theorems which have already been demonstrated, that make up the reasoning, and they would sustain it even if there were no diagram.[44]

Leibniz holds that the singular elements introduced through reliance on a diagram are inessential to a proof or derivation of the general propositions of geometry. Frege's passage does not squarely advocate Leibniz' position. But Frege seems to be explaining away the elements of singularity in his conception of pure intuition in order to avoid acknowledging that the general truths of geometry are derivative in any way from singular elements in intuition. This direction of thought about (pure) geometry seems to me reasonable and plausible.

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[44] Leibniz (1705; 1765; 1989) IV, i, 360-1.

But it is questionable whether Frege's view is really compatible with Kant's.

Kant sees himself as fundamentally at odds with Leibniz about geometry. He takes the role of pure intuition in geometry to be to produce an irreducibly singular element into mathematical understanding, reasoning, and justification. The problem for making these comparisons cleanly is that Kant's own view, though developed in great detail and subtlety, is not entirely clear or agreed upon.

I shall, however, sketch my view of it. Kant takes pure intuition in geometry to be intuitions of determinate objects. The objects of intuition are particulars, such as line-drawings, or even possible line-drawings, in pure geometrical intuition--pure imagination. (They can also be carried out in empirical intuition, on paper; but only non-empirical formal aspects of the empirical intuition play any role in mathematical understanding, reasoning, and justification.) From these objects one abstracts objects of a more general kind--"the triangle", for example--which are the objects of mathematical reasoning.[45] These latter objects are forms within the structure of space or time--on Kant's idealist view, forms of spatio-temporal intuition itself.

Theoretical cognition for Kant is fundamentally cognition of objects. Kant thought that pure mathematics has objects, and that those objects are not contingent, empirical objects.[46] "Determination" (<u>Bestimmung</u>) is a fundamental term in Kant's epistemology. Objects of successful theoretical cognition--the sort yielded in geometry--are necessarily determinate, or objects of determinate concepts, specific, non-vague concepts. They are abstracted from determinate particulars that are referents of pure intuition. The abstracted objects are determinate formal objects--spatial shapes, like triangles, and lines, planes, volumes. They form the subject matter of Euclidean geometry. The principles of geometry are about these objects. And thoughts about them are supported and guided by pure intuition about particular instances of these determinate objects. The role of intuition, hence the role of representation of <u>particulars</u>, is ineliminable from Kant's account of our understanding and warrant for pure geometry.

A passage in Kant that is comparable to the passages in Frege and Leibniz that we have just quoted is as follows:

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[45] Kant (1781, 1787) A713-4/B741-2; A723/B751.

[46] The point is denied in Friedman (1992) chapters 1 and 2. There are, however, numerous passages in which Kant makes it clear that he believes that pure mathematics has objects which are <u>not</u> the empirical objects experienced in space and time. For one such passage, see Kant (1781, 1787) A723/B751. I will develop these points in some detail in future work on Kant.

> Mathematical cognition [is reason-cognition out of] the construction of concepts. To construct a concept means to exhibit the intuition corresponding to it. For construction of a concept therefore a non-empirical intuition is required, which consequently as intuition is a single object (<u>einzelness Objekt</u>), but nonetheless, as the construction of a concept (of a general representation), it [the intuition] must express in the representation general [or universal] validity (<u>Allgemeingultigkeit</u>) for all possible intuitions, which belong under the same concept. Thus I construct a triangle by exhibiting the object corresponding to this concept, either through mere imagination in pure intuition, or in accordance therewith also on paper through empirical intuition, but in both cases purely apriori, without having had to borrow the pattern for it from any experience. The single drawn figure is empirical, yet it serves to express the concept without impairing its universality (<u>Allgemeinheity</u>); for in the case of this empirical intuition we look only at the action of the construction of the concept, to which [concept] many determinations [<u>Bestimmungen</u>]--for example, the magnitude of the sides and angles--are completely indifferent, and therefore we abstract from these differences, which do not alter the concept of triangle....mathematical cognition [considers] the general in the particular (<u>Besonderen</u>), in fact even in the individual (<u>Einzelnen</u>), although still apriori and by means of reason, so that just as this individual is determined under certain general conditions of construction, the object of the concept, to which this individual corresponds only as its schema, must be thought as universally (<u>allgemein</u>) determined.[47]

Frege's claim that "the intuited points, lines, planes are really not particular (<u>besondern</u>) at all" is definitely not compatible with Kant's view. Kant maintains that the referents of intuition are always particular or singular.[48] He takes the singularity of the intuition to be essential to the normative, justificational account of mathematical cognition. He takes abstraction from certain particularities inherent in the single object presented in pure

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[47] Kant (1781, 1787) A713-4/B741-2. The translation "we look at" and "we abstract from" is necessary for smooth rendering in English, but the German uses an impersonal passive construction in both cases.

[48] Actually for Kant the immediate referents of intuitions are property instances or mark-instances had by particular objects. And objects include parts of space and time as well as physical objects. But these are subtleties that we need not go into here.

intuition (or even in empirical intuition) to be necessary to understanding the mathematical concept (the general concept, triangle) and to doing pure geometry. But the singularity of the intuition is irreducibly part of the justification of mathematical cognition.

Frege explains the general validity of geometrical truths by maintaining that the particularity of pure intuition is only apparent. They can therefore "count as representatives of the whole of their kind". Like Kant, he sees the particulars as serving as representatives or stand-ins for more general features. He does not explain what role the singular aspects of intuition play in the process. But unlike Kant, he appears to be committed to thinking that they play no role in mathematical justification. This would explain his departure from Kantian doctrine in his claim that the intuited lines and so forth are not really particular at all. Unlike Kant, Frege is not interested in the particularity of mental acts in his explanation; this is a sign of his lack of commitment to Kantian idealism. He sees intuition as presenting typical geometrical structures which have no intrinsic individuality.

Kant explains the general validity of geometrical truths by maintaining that the particularity is genuine and ineliminable but is <u>used</u> as a schema. One abstracts from particular elements of the objects of intuition in forming a general object of the geometrical concept (and geometrical principle).

Like Frege, Kant does not make completely clear the role of the particular in warranting and guiding universal principles and inferential transitions. He seems to think that the particularistic elements in mathematical reasoning ground it in particular elements of space and time that reveal mathematical structures with maximum concreteness, and thus safeguard mathematical reasoning from the dangers that even transcendental philosophy is faced with. Kant seemed to think that mathematics' concern with particularity helps explain its certainty. But it <u>is</u> clear that he thought that the role of the particular is not to be explained away or seen as merely apparent. It is hard to escape the view that for Kant, in contrast to Frege, synthetic apriori propositions in geometry are grounded not in general propositions but in possible or actual particularistic judgments that are guided and supported by intuitions about particular, determinate objects of pure geometrical intuition. Although there are ways of understanding Frege's own view so as to render it internally consistent, and even sound, it is doubtful that it is consistent with Kant's.

Frege is aware of a need to discount the role of the particular, individual, or singular in geometrical warrant. If the general propositions rested, justificationally, on singular propositions, they could not be apriori in his sense.

Kant holds that the principles of geometry are strictly general or universally valid. He thinks that the basic principles are in the form of generalizations. But he does not hold that the root of geometrical warrant--the apriority of geometry--lies in generality.

The synthetic apriori axioms--and the inferential transitions--in pure geometry rest on non-general representations, pure intuitions. His examples of pure intuition supplementing our conceptions to yield warranted belief commonly involve propositions used singularly about particular geometrical constructions in Euclidean space.[49]

Kant claims that the successive synthesis of the productive imagination in the generation of figures--a process of singular representation--is the basis of axioms and inferences in Euclidean geometry. Although the axioms are general, their warrant does not rest on general propositions or general thoughts alone.[50]

There is a way of construing Frege's introduction of the notion of apriority that would reconcile his view with Kant's. Recall that Frege writes: "If...it is possible to derive the proof purely from general laws, which themselves neither need nor admit of proof, then the truth is apriori." (FA, section 3.) Geometrical proof, in the modern sense of "proof", starts with geometrical axioms. These are general. Thus for Frege "proof" in geometry rests on general truths, axioms. One might hold that Kant realized as well as anyone that geometrical proofs begin with the axioms. On his own view, the axioms are general (universally quantified). Thus interpreted, there is no disagreement.

What makes this resolution unsatisfying to me is that neither Frege nor Kant utilized precisely this modern notion of proof. For Frege, proof is canonical justification. The axioms are, on his view,

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[49] Kant (1781, 1787) A220-1/B267-8; A234/B287.

[50] Of course, in his theory of arithmetic, Kant denies that arithmetical propositions are derivable from axioms--hence from anything general--at all. He seems to regard the singular arithmetical operations and equations as basic. Cf. Kant (1781, 1787) A164-6/B204-6. Frege effectively criticized this extreme rejection of the role of axioms and proof in arithmetic Frege (1884) section 5. He is of course right in rejecting Kant's view that intuition enters into the justification of inferences in geometry and arithmetic. The issue of whether particularity is basic to mathematical justification is independent of whether justification of mathematical propositions (commonly) involves proof, and even of whether particularity enters the justification through non-conceptual intuition or directly from understanding. For a fine discussion of Kant's view of the role of intuition in inferences, see Friedman (1992) chapters 1 and 2. I believe that in supporting his sound view that Kant believed that intuition is necessary to mathematical inference, Friedman underplays the role of intuition in providing a basis for at least some of the axioms of Euclidean geometry. I think that Kant thought that intuitive constructions are as much a part of geometrical warrant and practice as commitment to the axioms is. Indeed the two go together.

general, self-evident, and in need of no warrant from anything further. For Kant the axioms and proofs in geometry are warranted through their relations to actual or possible line-drawings in pure intuition--thus through their relation to singular representations.[51] These representations must (to represent their objects at all) be conceptualized and backed by propositions or judgments in singular form.

So, Frege's notion of proof is one of canonical justification, not merely deductive sequences of thoughts. And on Kant's view axioms and proofs in geometry require warrant from pure intuition, which is essentially a faculty of singular representation. Unlike Frege, Kant is not wedded to a view of apriority that takes it to be founded in generality.[52] For Kant, synthetic apriori cognition is cognition that is grounded in the particular. For Kant the use of pure intuition is an integral part of geometrical practice and the mathematical understanding of the axioms and inferences themselves. Thus insofar as it is possible to compare like to like--Frege's epistemological conception of proof with Kant's conception of justificational reasoning within geometry--, the views of the two epistemologies appear quite different.

As I have emphasized, Frege leaves it unclear exactly what role intuition plays. But he implicitly denies a basic Kantian doctrine in holding that the objects of intuition are either not particular, or not fundamental to warrant in geometry. His picture of the role of particular elements in intuition seems in this respect to be more Leibnizean than Kantian. There is no evident room on his view to give intuition (as a singular representation) a warranting role.

I believe that Frege's verbal agreement with Kant about geometry is thus misleading. Frege accepts the language of Kant's doctrine of pure intuition--as applied to geometry. But it is doubtful that he can consistently accept all that Kant intends by this doctrine, and maintain the centrality of generality in his conception of apriority. Frege's Leibnizean conception of apriority takes generality of

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[51] Ultimately for Kant the warrant presupposes the point that space is a form of our intuition of physical objects. Cf. Kant (1781, 1787) A46-8/B64-6; B147. Hence the warrant for geometry (and indeed all of mathematics) depends on the alleged fact that its applicability to the world of experience is guaranteed through its having as its subject matter the forms of our experience. This is part of Kant's "transcendental deduction" of the objectivity of mathematics. I have little sympathy for this side of Kant's view, which in large part depends on his transcendental idealism.

[52] In fact, he contrasts apriori cognition in mathematics with apriori cognition in philosophy by maintaining that the central role of particularity in the justification of mathematical cognition. Cf. Kant (1781, 1787) A164/B204; A713-5/B741-3.

justificational starting point to be fundamental. He uses Kant's terminology of pure intuition, but he divests it of any commitment to referential singularity or reference to particulars, at least in its role in grounding geometrical principles. He retains Kant's view that intuition is essentially a non-rational (non-logical) faculty, thus appealing to intuition in order to explain his non-logicist, non-Leibnizean view of geometry. In this way he holds together a Leibnizean conception of apriority with a Kantian rejection of logicism about geometry. The fact that Frege provides a less detailed account of geometry, and less full explication of his term "intuition", than Kant does, is explained by Frege's preoccupation with the mathematics of number.

There is a further aspect of Frege's account of intuition in geometry that renders it very different from Kant's. Kant takes intuition to be a type of <u>objective</u> representation.[53] Frege holds that intuition is not objective. In fact, he explains objectivity partly in terms of independence from intuition, which he regards as essentially subjective (<u>FA</u>, section 26). In this passage, Frege makes his notorious claim that what is intuitable is not communicable. He sets out the thought experiment according to which what one being intuits as a plane another intuits as a point. He holds that since they can agree on geometrical principles (despite their subjective differences), their agreement is about something objective--about spatial structures that are subject to laws. Here again, it appears that particularistic aspects of intuition play no substantive role in Frege's account of the warrant for believing geometrical principles.

This doctrine of the subjectivity of spatial intuition is certainly not Kantian. Indeed, Kant characterizes intuition as an objective representation, in explicit contrast with subjective representations (sensations).[54] It is true that from a transcendental point of view, Kant regards space itself and hence pure apriori intuition as a form of our "subjective" constitution.[55] This is part of Kant's transcendental deduction of the objectivity of geometry. Kant thinks that only because, from the transcendental point of view, space, geometry, and apriori intuition are <u>all</u> to be construed idealistically as forms of the subject, can one account for the objectivity of apriori principles--and indeed the objectivity of pure intuition--in geometry about space. From the "empirical point of view"--the point of view of the practice of ordinary science and mathematics--, apriori intuition, geometrical principles, and space itself are all objectively valid and in no way confined to

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[53] Kant (1781, 1787) A320/B376-7.

[54] Kant (1781, 1787) A320/B377.

[55] Kant (1781, 1787) A48/B65.

individuals' subjectivity.

Frege appears to have thought that the ability of mathematicians to produce logically coherent non-standard geometries shows that one can conceive (though not imagine or intuit) the falsity of Euclidean geometry. He thought that our grasp of the self-evidence of the axioms of Euclidean geometry depends on some non-rational, or at least non-logical, capacity that he termed "intuition". The elements intuited that are captured by the axioms are common to all--and in fact can be grasped in thought even by subjects whose subjective intuitions differ from ours.[56] So particularistic aspects of the intuitions seem to play no role in their warranting the axioms.

Frege calls Euclidean axioms self-evident. This view is in some tension with his appeal to intuition as ground for the axioms. The warrant ("evidence") for believing the axioms seems not to rest purely in the senses of the axioms themselves. At least, one can apparently conceive of them as being false if one abstracts from spatial intuition. So the notion of self-evidence must be understood to include support from capacities whose deliverances is not entirely assimilated into the senses of the axioms themselves, or at least provides a support for them that is needed as supplement to any conceptual grasp of them that would abstract from such support.[57] Perhaps general features associated with what mathematicians intuit, but only general features, play a role in warranting the axioms.

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[56] This explication is well-expressed by Dummett (1982) p. 250. I believe also that Dummett is correct in arguing that there is substantial evidence against the view that Frege accepted Kantian idealism about space. For an excellent, general discussion of Frege's views on geometry, see Tappenden (1995).

[57] It is not entirely clear to me what Frege, in his mature post-Foundations work, thought the relation between intuition and the senses of geometrical propositions is. The subjective elements in intuition are surely not part of the senses. Whether he thought that in conceiving non-Euclidean geometries and regarding them as logically consistent yet incompatible with Euclidean geometry, we give different senses to the key terms ("straight") or give the same sense but somehow abstract from intuitive support is not clear to me. Frege seems to have thought that sometimes intuitions are used in symbolic ways, as representations of something other than what is intuited, in geometrical reasoning. For example, in discussing generalizations of geometry beyond Euclidean space to a space of four dimensions, Frege says that intuition is not taken for what it is but as symbolic for something else (Frege (1884) section 14). He may have seen the same sort of process as involved in conceiving Euclidean geometry false in the context of reasoning within non-Euclidean geometry. This is a matter that invites further investigation.

Both Kant and Frege held that Euclidean geometry yields apriori knowledge of physical space. As noted, this view is now untenable. What remains philosophically interesting is the epistemology of pure geometry. Warrant for mathematicians' belief in pure geometry seems to be apriori. Understanding the axioms seems sufficient to believe them. But what does such understanding consist in? Geometrical concepts appear to depend in some way on a spatial ability. Although one can translate geometrical propositions into algebraic ones and produce equivalent models, the meaning of the geometrical propositions seems to me to be thereby lost. Pure geometry has some spatial content, even if it involves abstraction from the exact empirical structure of physical space. Perhaps there is something in common to all legitimate spatial notions that any pure geometry makes use of. Whether the role for a spatial ability in our warrant for believing them is particularistic and non-conceptual--as Kant claims--or fully general and conceptual--as Leibniz, and seemingly Frege, believe--seems to me to invite further investigation.

I believe that Kant is likely to be right about the dependence of our understanding of pure geometries on our representation of spatial properties through sensory, non-rational capacities. Frege appears to have sided with Kant on this matter. I think that Kant is probably wrong in holding that a non-conceptual capacity, pure intuition, plays a warranting role in geometrical understanding much less geometrical inference. Leibniz' view of warrant as deduction from basic (conceptually) understood truths of pure geometry seems closer to a sound modern mathematical epistemology. Like Kant, Frege appears to give pure intuition a role in warranting at least belief in the axioms of geometry. (I know of no evidence that Frege agreed with Kant that intuition is essential to warranting geometrical inference.) But Frege gives pure intuition a role in geometrical warrant only after removing the key Kantian feature of singularity of reference from this role. Moreover, Frege's view of the relation between the role of intuition in geometrical warrant and the alleged subjective character of intuition is left unclear.

It seems to me that conceptual understanding of the axioms of the various pure geometries suffices to warrant one in believing those axioms, as propositions in pure mathematics. Intuition in the Kantian sense seems to play a role in the fixing of geometrical content, but not in the warrant for believing the axioms or rules of inference.

V.

I turn finally to the application of Frege's account of apriority to arithmetic. It is, of course, central to Frege's logicist project that truths about the numbers--which Frege certainly regarded as particular, determinate, formal objects (e.g. FA, sections 13, 18)--are derivative from general logical truths. The attempt to extract the existence and properties of particular objects from general principles centers, unfortunately, in Frege's defective Axiom V. There is a wide range of difficult issues here, and I cannot engage them seriously in this essay. But I want to broach, very briefly, some further points regarding Frege's characterization of apriority.

Suppose that Frege is mistaken, and arithmetic is not derivable in an epistemically fruitful way from purely general truths. Suppose that arithmetic has the form that it appears to have--a form that includes primitive singular intentional contents or propositions. For example, in the Peano axiomatization, arithmetic seems primitively to involve the thought that 0 is a number. And in normal arithmetical thinking we seem to know intentional contents that have singular form (0 + 1 = 1, for example) without deriving them from general ones. If some such knowledge is primitive--underived from general principles--, then it counts as aposteriori on Frege's characterization. This would surely be a defect of the characterization. The knowledge does not seem to rest on anything other than arithmetic understanding. This seems to be intellectual understanding. The justification of the knowledge does not involve sense experience in any way. Even though the knowledge does not seem to rest on pure sensible intuition, or on anything having essentially to do with perceptual capacities, it may be irreducibly singular.[58] Indeed it seems to be irreducibly singular from an epistemic point of view, regardless of whether it concerns (as it appears to) abstract but particular objects. At any rate, the failure of Frege's logicism gives one reason to worry whether apriority and generality coincide, even in the case of arithmetic. It seems to me, even after a century of reductive attempts, that we need a deeper investigation into the epistemology of arithmetic.

I think that from an epistemological perspective, arithmetic should be distinguished from set theory, second order logic, and various other parts of logic and mathematics. The enormous mathematical interest of the logicist project, and other reductive enterprises that have dominated this century, should not be allowed to obscure the fact that our understanding and hence our mode of knowing these other theories is different from our understanding of arithmetic. It seems to me even that the typical Peano formulation of arithmetic in terms of the successor function is epistemologically different from the formulation in terms of Arabic numerals on a base ten, which most of us learned first. Mathematical equivalence does not entail sameness of sense (in Frege's sense), and hence sameness of cognitive mode of presentation.

VI.

[58] In fact, our knowledge of set theory, while apriori, also seems to make primitive reference to particular sets, as noted earlier. Whereas Frege blamed set theory, rejecting it altogether, I am inclined to fault Frege's conception of apriority.

It is time to summarize. Frege's characterization of apriority in terms of generality is a mischaracterization. Apriority bears an essential connection to justificational independence from experience. In modern times, "experience" has come to mean sense experience. But Frege's characterization raises fundamental questions about the relation between apriority and generality. Frege followed a Leibnizean conception that assumed a close coincidence between the two notions.

If one thinks of experience sufficiently broadly (so as to include "intellectual experience" not just sense experience), some of the pressure against the coincidence can be dissipated. Such a conception may have been one of Leibniz' conceptions of experience, and the associated conception of apriority may therefore have been Leibnizean as well. Such a conception could treat the instances of the cogito and other token, indexically based, self-evident truths as aposteriori. This is because the conception construes apriority in a way that excludes from the apriori even justificational dependence on purely intellectual "experience". Given a Kantian conception of apriority, which is more in line with the dominant modern conception, self-knowledge and knowledge of certain other indexical-involving truths can be apriori. For warrant seems to derive purely from intellectual understanding. It in no way rests on sense-perception.

Problems with geometry and arithmetic remain. Leibniz, Kant, and Frege all maintained that geometry and arithmetic are apriori. If the position is carefully confined to pure geometry, it seems highly plausible. I believe, however, that we do not understand very well the role of spatial abilities in the content and justification of pure geometries. So I think that it is not fully clear whether justification in pure geometry rests on purely general propositions, although it seems to me likely that it does. The case of arithmetic is, I think, more serious as a possible counterexample to the claim of a coincidence between apriority and the primacy of generalizations in canonical justification. For arithmetic is apparently committed to basic truths in singular form, in its most natural and straightforward formulations.

I think that Frege is right to reject Kant's claim that the deliverances of a non-conceptual faculty, pure intuition, is justificationally basic in the warrant for arithmetic. But Kant may nevertheless have been right to hold that although cognition of arithmetic is apriori, cognition (or propositions) in singular form can be justificationally basic. One's justification derives from an understanding that encompasses singular intentional contents. On such a view, some apriority would be non-logical, and would not derive purely from general principles of pure reason. In arithmetic apriori knowledge would derive from intellectual, non-sense-perceptual understanding of necessary, non-context-dependent, singular intentional contents. I think that we should investigate in more depth the innovation that Kant offered: apriority that does not rest on logical or other general principles. I recommend doing so without assuming that apriori theoretical cognition must be constrained, as Kant insisted, by relation to sensibility. I recommend doing so without presuming that we must invoke Kant's notion of pure sensible intuition. I believe that we can follow Leibniz and Frege in avoiding essential reliance on pure intuition in arithmetic, without following them in insisting that generality lies at the base of all apriori warrant. Kant's conception of underived, singular understanding which is nevertheless apriori seems to me worth pursuing.

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